arXiv:1502.03277v1 [math.AG] 11 Feb 2015

$A+B$ THEORY IN CONIFOLD TRANSITIONS FOR CALABI–YAU THREEFOLDS

YUAN-PIN LEE, HUI-WEN LIN, AND CHIN-LUNG WANG

ABSTRACT. For projective conifold transitions between Calabi-Yau threefolds X and Y, with X close to Y in the moduli, we show that the combined information provided by the A model (Gromov–Witten theory in all genera) and B model (variation of Hodge structures) on X determines the corresponding combined information on Y, and vice versa.

CONTENTS

0. INTRODUCTION

0.1. **Statements of results and idea of proofs.** Let X be a smooth projective 3-fold. A (projective) conifold transition $X \nearrow Y$ is a projective degeneration

$$\pi : \mathfrak{X} \to \Delta$$

of X to a singular variety $\bar{X} = \mathfrak{X}_0$ with a finite number of ordinary double points (abbreviated as ODPs or nodes) $p_1, \cdots, p_k$, locally analytically defined by the equation

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0,$$

followed by a projective small resolution

$$\psi : Y \to \bar{X}.$$

In the process of complex degeneration from X to $\bar{X}$, k vanishing cycles $S_i \cong S^3$ with trivial normal bundle collapse to nodes p_i. In the process of "Kähler degeneration" from Y to $\bar{X}$, the exceptional loci of ψ above each p_i

is a smooth rational curve $C_i \cong \mathbb{P}^1$ with $N_{C_i/Y} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. (See Section 1 for details.) We write $Y \searrow X$ for the reverse process.

Notice that ψ is a crepant resolution and π is a finite distance degeneration with respect to the quasi-Hodge metric [37, 38]. A transition of this type (in all dimensions) is called an extremal transition. All known Calabi–Yau 3-folds with the same fundamental group are connected through extremal transitions, of which conifold transitions are the simplest kind. It therefore makes sense to start the investigation with conifold transitions. In this paper we mainly consider conifold transitions among *projective Calabi–Yau threefolds*.

We start by studying the changes of the so-called A model and B model under a general projective conifold transition. In the scope of this paper, the A model is the Gromov–Witten theory of all genera; the B model is the variation of Hodge structures (VHS), which is in a sense only the genus zero part of the quantum B model. A conifold transition, or more generally an extremal transition, can be regarded as a finite distance B model degeneration followed by an inverse of a finite distance A model degeneration. In contrast to the usual *birational K* equivalence, an extremal transition may be considered as a *generalized K* equivalence in the sense that ψ is crepant and the degenerating family π preserves sections of canonical bundles.

In general, the conditions for the existence of projective conifold transitions is an unsolved problem except in the case of Calabi–Yau 3-folds, for which we have fairly complete understanding. For the inverse conifold transition $Y \searrow X$, a celebrated theorem of Friedman [7] (also Kawamata [14] and Tian [36]) states that a small contraction $Y \to \bar{X}$ can be smoothed if and only if there is a totally nontrivial relation between the exceptional curves. (Friedman's theorem was inspired by Clemens's earlier work [4].) That is, there exist constants $a_i \neq 0$ for all $i = 1, \ldots, k$ such that $\sum_{i=1}^k a_i[C_i] = 0$. These are relations among curves $[C_i]$'s in the kernel of $H_2(Y)_\mathbb{Z} \to H_2(X)_\mathbb{Z}$. Let μ be the number of independent relations and let $A \in M_{k\times\mu}(\mathbb{Z})$ be the relation matrix for C_i's. Therefore, the dimension of $H^2(Y)/H^2(X)$ is $k - \mu$. Conversely, Smith, Thomas and Yau proved dual statement in [34], asserting that for a conifold transition $X \nearrow Y$ the k vanishing 3-spheres S_i must satisfy a totally nontrivial relation $\sum_{i=1}^k b_i[S_i] = 0$ with $b_i \neq 0$ for all i. They are relations among the vanishing cycles $[S_i]$'s in $V_\mathbb{Z} := \ker(H_3(X)_\mathbb{Z} \to H_3(\bar{X})_\mathbb{Z})$. Similarly, let ρ be the number of independent relations and $B \in M_{k\times\rho}(\mathbb{Z})$ be the relation matrix for S_i's. Thus dimension of V is $k - \rho$.

The relation matrices A and B are defined for general conifold transitions regardless the Calabi–Yau assumption on X and Y. It turns out that $\mu + \rho = k$ and the following exact sequence holds.

Theorem 0.1 (= Theorem 2.9). *Under a conifold transition $X \nearrow Y$ of smooth projective threefolds, we have*

$$0 \to H^2(Y)/H^2(X) \xrightarrow{B} \mathbb{C}^k \xrightarrow{A^t} V \to 0$$

as a exact sequence of weight two Hodge structures.

We interpret this as a partial exchange of topological information between the *excess A model* of Y over X (in terms of $H^2(Y)/H^2(X)$) and the *excess B model* of X over Y in terms of V.

The main goal of this paper is to study the changes of A and B models under a projective conifold transition of Calabi–Yau 3-folds and its inverse. We remark that the Kuranishi space $\mathcal{M}_{\bar{X}}$ is smooth due to the unobstructedness result of Ran, Kawamata and Bogomolov–Tian–Todorov. Our main result is the following theorem, stated in a *imprecise form* below.

Theorem 0.2. *Let $X \nearrow Y$ be a projective conifold transition of Calabi–Yau threefolds such that $[X]$ is a nearby point of $[\bar{X}]$ in $\mathcal{M}_{\bar{X}}$. Then*

$$(A + B) \text{ theory of } X \Longleftrightarrow (A + B) \text{ theory of } Y.$$

More precisely,

(1) *$A(X)$ is a sub-theory of $A(Y)$.*
(2) *$B(Y)$ is a sub-theory of $B(X)$.*
(3) *$A(Y)$ can be reconstructed from a refined A model of $X^\circ := X \setminus \bigcup_{i=1}^k S_i$ "linked" by the vanishing spheres.*
(4) *$B(X)$ can be reconstructed from a refined B model of $Y^\circ := Y \setminus \bigcup_{i=1}^k C_i$ "linked" by the exceptional curves.*

The meaning of these slightly obscure statements can not be made completely precise in the limited space here, but will take the entire paper to spell them out. Nevertheless, we will attempt to give a brief explanation below.

(1) is due to Li–Ruan [22]. Even though transitions for A model were intensively studied in the physics literature since 1990's, the mathematical study started in [22]. In fact, it follows from their degeneration formula (cf. Proposition 3.1) that Gromov–Witten invariants on X can be reconstructed from those on Y. This gives (1).

For (2), we note that there are natural identifications of $\mathcal{M}_Y$ with the boundary of $\mathcal{M}_{\bar{X}}$ consisting of equisingular deformations, and $\mathcal{M}_X$ with $\mathcal{M}_{\bar{X}} \setminus \mathfrak{D}$ where the discriminant locus $\mathfrak{D}$ is a *central hyperplane arrangement* with axis $\mathcal{M}_Y$ (cf. Section 4, especially Section 4.3.3). Therefore, the VHS associated to Y can be considered as a sub-system of VHS (with logarithmic singularity) associated to $\bar{X}$, which is a regular singular extension of the VHS associated to X.

With (3), we first have to introduce the "linking data" of the holomorphic curves in X°, which not only records the curve classes but also how the curve links with the vanishing spheres $\bigcup_i S_i$. The linking data on X can be identified with the curve classes in Y by $H_2(X^\circ) \cong H_2(Y)$ (cf. Definition 5.3 and (5.4)). We then proceed to show, by the degeneration argument, that the virtual class of moduli spaces of stable maps to X° is naturally a disjoint union of pieces labeled by elements of the linking data (c.f. Proposition 5.9): Given β a curve class in X, we can associate to it a set of curve classes γ on Y, called the liftings of β, so that there is a decomposition

$$[\overline{M}(X,\beta)]^{\text{virt}} = \coprod_{\gamma \in H_2(X^\circ)} [\overline{M}(X,\gamma)]^{\text{virt}} \sim \coprod_{\gamma \in H_2(Y)} [\overline{M}(Y,\gamma)]^{\text{virt}}.$$

Furthermore, the Gromov–Witten invariants of the curve class γ in Y is the same as the numbers produced by the component of the virtual class on X labeled by the corresponding linking data (c.f. Proposition 5.9). Thus, the refined A model is really the "linked A model" and the linked A model on X is equivalent to the (usual) A model of Y (for non-extremal curves classes) in all genera. Note that here the vanishing cycles from $B(X)$ plays a key role in reconstructing $A(Y)$.

For (4), the goal is to reconstruct VHS on $\mathcal{M}_X$ from VHS on $\mathcal{M}_Y$ and $A(Y)$. The deformation of $\bar{X}$ is unobstructed. Moreover it is well known that $\mathrm{Def}(\bar{X}) \cong H^1(Y^\circ, T_{Y^\circ})$. Even though geometrically the deformation of Y° is obstructed (in the direction transversal to $\mathcal{M}_Y$), there is a first order deformation parameterized by $H^1(Y^\circ, T_{Y^\circ})$ which gives enough initial condition to uniquely determine the degeneration of Hodge bundles on $\mathcal{M}_{\bar{X}}$ near $\mathcal{M}_Y$. The most technical result needed in this process is a short exact sequence

$$0 \to V \to H^3(X) \to H^3(Y^\circ) \to 0$$

which connects the limiting mixed Hodge structure (MHS) of Schmid on $H^3(X)$ and the canonical MHS of Deligne on $H^3(Y^\circ)$ (c.f. Proposition 6.1). Together with the monodromy data associated to the ODPs, which is encoded in the relation matrix A of the extremal rays on Y, we will be able to determine the VHS on $\mathcal{M}_X$ near $\mathcal{M}_Y$. In the process, we need a slight extension of Schmid's nilpotent orbit theorem [32] to degenerations with certain non-normal crossing discriminant loci.

Consider a degeneration of polarized Hodge structures over $\Delta^\mu \times M$ with discriminant locus $\mathfrak{D} = \bigcup_{i=1}^k D^i$ being a central hyperplane arrangement with axis M. Let $N^{(i)}$ be the nilpotent monodromy around the hyperplane $D^i = Z(w_i)$ and suppose that the monodromy group Γ generated by $N^{(i)}$'s is *abelian*. Let D denote the period domain and $\check{D}$ its compact dual.

We prove in Theorem 4.15 that the period map

$$\phi : \Delta^\mu \times M \setminus \mathfrak{D} \to D/\Gamma$$

takes the following form

$$\phi(r,s) = \exp\left(\sum_{i=1}^k \frac{\log w_i}{2\pi\sqrt{-1}} N^{(i)}\right) \psi(r,s),$$

where $\psi : \Delta^\mu \times M \to \check{D}$ is holomorphic and horizontal.

The fact that the monodromy group Γ is abelian for conifold degenerations follows from the Picard–Lefschetz formula easily (c.f. (4.5) in Section 4). In particular it applies to degenerations over $\mathcal{M}_{\bar{X}}$ associated to conifold transitions $X \nearrow Y$ of Calabi–Yau 3-folds through $\bar{X}$. In general, the abelian constraint is automatic if $\mu \geq 3$ (see Remark 4.16).

In the proof of Theorem 0.2 (4) (Section 6.3), it turns out that the natural coordinates $r_1, \cdots, r_\mu$ on $\mathcal{M}_{\bar{X}}$ in the directions transversal to $\mathcal{M}_Y$ are given by periods of independent vanishing cycles $\Gamma_1, \cdots, \Gamma_\mu$ (a basis of V):

$$r_j = \int_{\Gamma_j} \Omega,$$

where Ω is a relative holomorphic 3-form over $\mathcal{M}_{\bar{X}}$. The horizontal map ψ corresponds to the refined B model on Y°, and the linear forms

$$w_i = \int_{S_i} \Omega = a_{i1} r_1 + \cdots + a_{i\mu} r_\mu, \qquad 1 \leq i \leq k$$

correspond to the $k \times \mu$ relation matrix $A = (a_{ij})$. Thus the above factorization of ϕ gives the precise description of the "linking" of $B(Y^\circ)$ by the exceptional curves C_i's (viewed as data from $A(Y)$). This completes the outline of the proof of Theorem 0.2.

The following comparison of proofs of (3) and (4) might be helpful.

In the proof of (3), we calculate the Gromov–Witten invariants associated to the extremal rays in Y, via the multiple cover formulas and relation matrix B of the vanishing spheres S_i's in X. As a consequence we determine the monodromy along the discriminant loci in the Kähler moduli (see Section 3.3). Combined with the refined A model on X° we then determine $A(Y)$. Notice that we actually identify all Gromov–Witten invariants without making use of the monodromy or the Dubrovin connection—though at the end we may write down a dual formulation as in Theorem 4.15.

In the proof of (4), the starting point is the Picard–Lefschetz monodromy and then the refined B model on Y°. Finally they are "linked" via the extended nilpotent orbit theorem (Theorem 4.15). In this approach the corresponding *invariants* are the so called *Yukawa couplings*, which turn out are derived as consequences (see Proposition 4.18 and Section 4.3.4) and are not used in the proof. Thus, while the general structures on both directions are similar, the technical details and logic of proof are different.

0.2. **Motivation and future plans.** All known examples of Calabi–Yau 3-folds of the same fundamental group are connected by extremal transitions, and many of them are indeed known to be connected by conifold transitions. The famous Reid's fantasy [29] suggests the possibility that in fact all of them are connected by conifold transitions. Therefore, in order to study A model or B model of any Calabi–Yau threefold one might "only" needs to study their changes under an extremal (or even conifold) transition and one simplest example, which is "easy". This work is meant to be the first general study in this direction.

Theorem 0.2 above can be interpreted as partial exchange of A and B models under a conifold transition. We hope to be able to answer the following intriguing question concerning with "global symmetries" on moduli spaces of Calabi–Yau 3-folds in the future: *Would this partial exchange of A and B models lead to "full exchange" when one connects a Calabi–Yau threefold to its mirror via a finite steps of extremal transitions? If so, what is the relation between this full exchange and the one induced by "mirror symmetry"?* In particular, the Fermat quintic and its mirror would be an excellent testing ground as their genus zero A model are both computed in [9, 24] and [19]. To this end, we need to devise a computationally effective way to achieve explicit determination. One missing piece of ingredients in this direction is a blowup formula in the Gromov–Witten theory for conifolds, which we are working on and have had some partial success [18]. [1] The reverse implication is not constructive either. It might be possible to explicitly construct the VHS of X from that of Y° via the logarithmic model of degenerating Hodge structure of Steenbrink [35] (and Clemens [5]). The details remain to be worked out.

More speculatively, the mutual determination of A and B models on X and Y leads us to surmise the possibility of a unified "$A + B$ model" which will be *invariant* under any extremal transition. For example, the string theory only predicts that Calabi–Yau threefolds form an important ingredient of our universe, but fails to tell us which Calabi–Yau threefold we should live in. Should the $A + B$ model be available and proven to be invariant under any extremal transition, there is no need to choose which universe to live in (at least for the worlds governed by the TQFT).

The first step of achieving this goal is to find a $\mathcal{D}$-module version of the basic exact sequence (Theorem 0.1). On V there is a natural flat connection given by the Gauss–Manin connection. $H^2(Y)/H^2(X)$ is naturally endowed with the Dubrovin connection. Therefore, it is not unreasonable to expect a $\mathcal{D}$-module lift of the basic exact sequence (c.f. Proposition 7.1), which may be heuristically interpreted as

$$\text{"excess } A \text{ theory"} + \text{"excess } B \text{ theory"} = \text{"trivial"}.$$

We hope to be able to "glued" the flat (log) connections of the excess theories to the Dubrovin connection on the A side and the Gauss–Manin connection on the B side. This will be a key step in constructing the speculative $A + B$ theory.

0.3. **Outline of the paper.** In Section 1 we review the basic geometry of a projective conifold transition.

[1] For (smooth) blowups with complete intersection centers, we have a fairly good solution in genus zero.

In Section 2, we compute the limiting mixed Hodge structures of the two semistable models associated to the conifold degeneration. Using ingredients from Hodge theory, we derive the basic exact sequence in Theorem 2.9.

Section 3 is devoted to some discussions on Gromov–Witten theory under a conifold transition. We explain Theorem 0.2 (1) in Section 3.1 and then concentrate on the genus zero Gromov–Witten theory associated to the exceptional curves (extremal rays) of $\psi : Y \to \bar{X}$.

In Section 4, we recall the relevant deformation theory of Calabi–Yau threefold conifolds and extend part of Bryant–Griffiths's study of periods of smooth Calabi–Yau threefolds to Calabi–Yau conifolds. In doing so, we prove an extension of the nilpotent orbit theorem where the discriminant loci is *not* a simple normal crossing divisor but a central hyperplane arrangement. Using it, we identify the singular part of the period map.

Section 5 finishes the proof of Theorem 0.2 (3). The major new construction in this section is the definition of the refined Gromov–Witten invariants on $X^\circ := X \setminus \bigcup_{i=1}^k S_i$. Together with ingredients on extremal ray invariants from Section 3 we complete the determination of $A(Y)$.

With Section 6 the proof Theorem 0.2 (4) is complete. The major theme in this section is to study the deformation theory on $Y^\circ := Y \setminus \bigcup_{i=1}^k C_i$. The resulting variations of mixed Hodge structures is what we called the refined B model on Y°. Together with the extended nilpotent orbit theorem we complete the determination of $B(X)$.

The paper is concluded in Section 7 by two remarks concerning our future plans on the $\mathcal{D}$-module lift of the basic exact sequence and effective methods to determine Gromov–Witten theory on Y in terms of X.

0.4. **Acknowledgements.** We are grateful to C.H. Clemens, C.-C. M. Liu, I. Smith, and R. Thomas, for discussions related to this project.

Y.-P. L.'s research is partially funded by the National Science Foundation. H.-W. Lin and C.-L. Wang are both supported by the Ministry of Science and Technology, Taiwan. We are grateful to Taida Institute of Mathematical Sciences (TIMS) for its generous and constant support which makes this long term collaboration possible.

1. Preliminaries of conifold transitions

In Sections 1–3, all discussions are for any projective conifold transition *without the Calabi–Yau condition*, unless otherwise specified. The Calabi–Yau condition is imposed in Sections 4–6.

1.1. Local geometry.

Definition 1.1. Let X be a smooth projective 3-fold. A (*projective*) *conifold transition* $X \nearrow Y$ is a projective degeneration

$$\pi : \mathfrak{X} \to \Delta$$

of X to a singular variety $\bar{X} = \mathfrak{X}_0$ with a finite number of ordinary double points (ODPs, nodes, A_1 singularities) $p_i, \cdots, p_k$, followed by a projective small resolution

$$\psi : Y \to \bar{X}.$$

We write $Y \searrow X$ for the inverse conifold transition.

Locally analytically, an ordinary double point is defined by the equation

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0, \tag{1.1}$$

or equivalently

$$uv - ws = 0.$$

The small resolution ψ can be achieved by blowing up the Weil divisor defined by $u = w = 0$ or by $u = s = 0$, these two choices differ by a flop.

Lemma 1.2. *The exceptional locus of ψ above each p_i is a smooth rational curve $C_i \cong \mathbb{P}^1$ with the normal bundles*

$$N_{C_i/Y} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1).$$

Topologically, $N_{C_i/Y}$ is a trivial rank 4 real vector bundle.

Proof. This follows from the above local description of blowing up. Away from the isolated singular points p_i's, the Weil divisors are Cartier and the blowups do nothing. Locally near p_i, the Weil divisor is generated by two functions u and w. The blowup $Y \subset \mathbb{A}^4 \times \mathbb{P}^1$ is defined by $z_0 v - z_1 s = 0$, in addition to $uv - ws = 0$ defining X, where $(z_0 : z_1)$ are the coordinates of $\mathbb{P}^1$. Namely we have

$$\frac{u}{w} = \frac{s}{v} = \frac{z_0}{z_1}.$$

It is now easy to see the exceptional locus near p_i is isomorphic to $\mathbb{P}^1$ and the normal bundle is as described (by the definition of $\mathcal{O}_{\mathbb{P}^1}(-1)$). It is topologically trivial since all $\mathbb{Z}/2$ Stiefel–Whitney classes w_k's are zero. ☐

Locally to each node $p = p_i \in \bar{X}$, the transition $X \nearrow Y$ can be considered as two different ways of "smoothing" the singularities in $\bar{X}$: deformation leads to X_t and small resolution leads to Y. Topologically, we have seen that the exceptional loci of ψ are $\coprod_{i=1}^k C_i$, a disjoint union of k 2-spheres. For the deformation, the classical results of Picard, Lefschetz and Milnor state that there are k vanishing 3-spheres $S_i \cong S^3$.

Lemma 1.3. *Topologically the normal bundle*

$$N_{S_i/X_t} \cong T^*_{S_i}$$

is a trivial rank 3 real vector bundle.

Proof. From the local description of the singularity (1.1), we have, after degree two base change, the local equation of the family near an ordinary double point:

$$\sum_{j=1}^{4} x_j^2 = t^2 = |t|^2 e^{2\sqrt{-1}\theta}.$$

With a simple change of variables $y_j = e^{\sqrt{-1}\theta} x_j$ for $j = 1, \ldots, 4$, the equation becomes

$$\sum_{j=1}^{4} y_j^2 = |t|^2 \tag{1.2}$$

Write y_j in terms of real coordinates $y_j = a_j + \sqrt{-1} b_j$, (1.2) becomes

$$|\vec{a}|^2 = |t|^2 + |\vec{b}|^2 \quad \text{and} \quad \vec{a} \cdot \vec{b} = 0, \tag{1.3}$$

where $\vec{a}$ and $\vec{b}$ are two vectors in $\mathbb{R}^4$. The set of solutions to (1.3) can be identified with T^*S_r with the bundle structure $T^*S_r \to S_r$ defined by

$$(\vec{a}, \vec{b}) \mapsto r\frac{\vec{a}}{|\vec{a}|} \in S_r$$

where S_r is the 3-sphere with radius $r = |t|$. The vanishing sphere can be chosen to be the real locus of the equation of (1.2). Therefore, N_{S_r/X_t} is naturally identified with the cotangent bundle T^*S_r, which is a trivial bundle since $S^3 \cong SU(2)$ is a Lie group. □

Remark 1.4. We see from the above description that the vanishing spheres are Lagrangian with respect to the natural symplectic structure on T^*S^3. A theorem of Seidel and Donaldson [33] states that this is true globally, namely the vanishing spheres can be chosen to be Lagrangian with respect to the symplectic structure coming from the Kähler structure of X_t.

By Lemma 1.2, the δ neighborhood of the vanishing 3-sphere S_r^3 in X_t is homeomorphic to trivial disc bundle $S_r^3 \times D_\delta^3$. By Lemma 1.2 the r neighborhood of the exceptional 2-sphere $C_i = S_\delta^2$ is $D_r^4 \times S_\delta^2$, where δ is the radius defined by $4\pi\delta^2 = \int_{C_i} \omega$ for the background Kähler metric ω. Therefore, we have the following conclusion.

Corollary 1.5. *On the topological level one can go between Y and X_t by surgery via*

$$\partial(S_r^3 \times D_\delta^3) = S_r^3 \times S_\delta^2 = \partial(D_r^4 \times S_\delta^2).$$

Remark 1.6 (Orientations on S^3). The two choices of orientations on S_r^3 induces two different surgeries. The resulting manifolds Y and Y' are in general not even homotopically equivalent. In the complex analytic setting the induced map $Y \dashrightarrow Y'$ is known as an ordinary (Atiyah) flop.

1.2. **Global topology.** Now we turn to the global topological constraint.

Lemma 1.7. *Define*

$$\mu := \tfrac{1}{2}(h^3(X) - h^3(Y))$$

and

$$\rho := h^2(Y) - h^2(X)$$

Then,

$$\mu + \rho = k. \tag{1.4}$$

Proof. The Euler numbers satisfy

$$\chi(X) - k\chi(S^3) = \chi(Y) - k\chi(S^2).$$

That is,

$$2 - 2h^1(X) + 2h^2(X) - h^3(X) = 2 - 2h^1(Y) + 2h^2(Y) - h^3(Y) - 2k.$$

By the above surgery argument we know that conifold transitions preserve the fundamental group. Therefore,

$$\tfrac{1}{2}(h^3(X) - h^3(Y)) + (h^2(Y) - h^2(X)) = k.$$

□

Remark 1.8. In the Calabi-Yau case, $\mu = h^{2,1}(X) - h^{2,1}(Y) = -\Delta h^{2,1}$ is the lose of complex moduli, and $\rho = h^{1,1}(Y) - h^{1,1}(X) = \Delta h^{1,1}$ is the gain of Kähler moduli. Thus (1.4) is really $\Delta(h^{1,1} - h^{2,1}) = k = \frac{1}{2}\Delta\chi$.

This might suggest the expression $A - B$ instead of $A + B$. We use the latter since it really means a *combined (A, B) theory*, with the interpretation that A corresponds to H^{ev} and B corresponds to H^{odd}.

In the next section, we will study the *Hodge-theoretic meaning* of this simple topological equality.

2. Hodge theory and the basic exact sequence

Convention. In this paper, unless otherwise specified, cohomology groups are over $\mathbb{Q}$ when only topological aspect (including weight filtration) is concerned; they are considered over $\mathbb{C}$ when the (mixed) Hodge-theoretic aspect is involved.

All equalities, whenever they make sense in the context of mixed Hodge structure (MHS), hold as equalities for MHS unless otherwise specified.

2.1. **Two semistable degenerations.** In order to apply Hodge-theoretic techniques on the degenerations, we factor the transition $X \nearrow Y$ as a composition of two semistable degenerations $\mathcal{X} \to \Delta$ and $\mathcal{Y} \to \Delta$.

The *complex degeneration*

$$f : \mathcal{X} \to \Delta \tag{2.1}$$

is the semistable reduction for $\mathfrak{X} \to \Delta$ obtained by a degree two base change $\mathfrak{X}' \to \Delta$ followed by the blow-up $\mathcal{X} \to \mathfrak{X}'$ of all the four dimensional nodes $p_i' \in \mathfrak{X}'$. The special fiber $\mathcal{X}_0 = \bigcup_{j=0}^k X_j$ is a simple normal crossing divisor with

$$\tilde{\psi} : X_0 \cong \tilde{Y} := \mathrm{Bl}_{\coprod_{i=1}^k \{p_i\}} \bar{X} \to \bar{X}$$

being the blow-up at the nodes and with

$$X_i = Q_i \cong Q \subset \mathbb{P}^4, \quad i = 1, \ldots, k$$

being quadric threefolds. Let $X^{[j]}$ be the disjoint union of $j+1$ intersections from X_i's. Then the only nontrivial terms are

$$X^{[0]} = \tilde{Y} \coprod_i Q_i \qquad \text{and} \qquad X^{[1]} = \coprod_i E_i$$

where

$$E_i = \tilde{Y} \cap Q_i \cong \mathbb{P}^1 \times \mathbb{P}^1$$

are the $\tilde{\psi}$ exceptional divisors. The semistable reduction f does not require the existence of a small resolution of $\mathfrak{X}_0$.

The *Kähler degeneration*

$$g : \mathcal{Y} \to \Delta \tag{2.2}$$

is simply the deformations to the normal cone

$$\mathcal{Y} = \mathrm{Bl}_{\coprod C_i \times \{0\}} Y \times \Delta \to \Delta.$$

The special fiber $\mathcal{Y}_0 = \bigcup_{j=0}^k Y_j$ with

$$\phi : Y_0 \cong \tilde{Y} := \mathrm{Bl}_{\coprod_{i=1}^k \{C_i\}} Y \to Y$$

being the blow-up along the curves C_i's and

$$Y_i = \tilde{E}_i \cong \tilde{E} = P_{\mathbb{P}^1}(\mathcal{O}(-1)^2 \oplus \mathcal{O}), \quad i = 1, \ldots, k.$$

In this case the only non-trivial terms for $Y^{[j]}$ are

$$Y^{[0]} = \tilde{Y} \coprod_i \tilde{E}_i \qquad \text{and} \qquad Y^{[1]} = \coprod_i E_i$$

where

$$E_i = \tilde{Y} \cap \tilde{E}_i$$

is now understood as the infinity divisor (or relative hyperplane section) of $\pi_i : \tilde{E}_i \to C_i \cong \mathbb{P}^1$.

2.2. **Mixed Hodge Structure and the Clemens–Schmid exact sequence.** We apply the Clemens–Schmid exact sequence to the above two semistable degenerations. A general reference for the background material here is [11]. We will mainly be interested in $H^{\leq 3}$, although the computation of $H^{>3}$ is similar.

2.2.1. The cohomology of the central fiber $H^*(\mathcal{X}_0)$, with its canonical mixed Hodge structure, is computed from the spectral sequence

$$E_1^{p,q}(\mathcal{X}_0) = H^q(X^{[p]})$$

with the differential $d_1 = \delta$ the combinatorial coboundary operator

$$\delta : H^q(X^{[p]}) \to H^q(X^{[p+1]}).$$

The spectral sequence degenerates at E_2 terms. The weight filtration on $H^*(\mathcal{X}_0)$ is induced from the following increasing filtration on the spectral sequence $W_m := \bigoplus_{q \le m} E^{*,q}$. Therefore,

$$\mathrm{Gr}_m^W(H^j) = E_2^{j-m,m}, \quad \mathrm{Gr}_m^W(H^j) = 0 \quad \text{for } m < 0 \text{ or } m > j.$$

Since $X^{[j]} \neq \emptyset$ only when $j = 0, 1$, we have

$$H^0 \cong E_2^{0,0}, \quad H^1 \cong E_2^{1,0} \oplus E_2^{0,1}, \quad H^2 \cong E_2^{1,1} \oplus E_2^{0,2}, \quad H^3 \cong E_2^{1,2} \oplus E_2^{0,3}.$$

The only weight 3 piece is $E_2^{0,3}$, which can be computed by

$$\delta : E_1^{0,3} = H^3(X^{[0]}) \longrightarrow E_1^{1,3} = H^3(X^{[1]}).$$

Since Q_i, $\tilde{E}_i$ and E_i have no odd cohomologies, $H^3(X^{[1]}) = 0$ and $H^3(X^{[1]}) = H^3(\tilde{Y})$. We have thus $E_2^{0,3} = H^3(\tilde{Y})$.

The weight 2 pieces, which is the most essential part, can be computed from the following map

$$H^2(X^{[0]}) = H^2(\tilde{Y}) \oplus \bigoplus_{i=1}^{k} H^2(Q_i) \xrightarrow{\delta_2} H^2(X^{[1]}) = \bigoplus_{i=1}^{k} H^2(E_i). \tag{2.3}$$

We have $E_2^{1,2} = \mathrm{cok}(\delta_2)$ and $E_2^{0,2} = \ker(\delta_2)$.

The weight 1 and weight 0 pieces can be similarly computed. For weight 1 pieces we have

$$E_2^{0,1} = H^1(X^{[0]}) = H^1(\tilde{Y}) \cong H^1(Y) \cong H^1(X),$$

and $E_2^{1,1} = 0$. The weight 0 pieces are computed from

$$\delta : H^0(X^{[0]}) \to H^0(X^{[1]})$$

and we have

$$E_2^{0,0} = H^0(\tilde{Y}) \cong H^0(Y) \cong H^0(X),$$

and $E_2^{1,0} = 0$.

We summarize these calculations in the following lemma.

Lemma 2.1.

$$\begin{aligned} H^3(\mathcal{X}_0) &\cong H^3(\tilde{Y}) \oplus \mathrm{cok}(\delta_2), \\ H^2(\mathcal{X}_0) &\cong \ker(\delta_2), \\ H^1(\mathcal{X}_0) &\cong H^1(\tilde{Y}) \cong H^1(Y) \cong H^1(X), \\ H^0(\mathcal{X}_0) &\cong H^0(\tilde{Y}) \cong H^0(Y) \cong H^0(X). \end{aligned}$$

In particular, $H^j(\mathcal{X}_0)$ is pure of weight j for $j \le 2$.

2.2.2. Here we give a dual formulation of (2.3) which will be useful later. Let ℓ, ℓ' be the line classes of the two rulings of $E \cong \mathbb{P}^1 \times \mathbb{P}^1$. Then $H^2(Q, \mathbb{Z})$ is generated by $e = [E]$ as a hyperplane class and $e|_E = \ell + \ell'$. The map δ_2 in (2.3) is then equivalent to

$$\bar{\delta}_2 : H^2(\tilde{Y}) \longrightarrow \bigoplus_{i=1}^{k} H^2(E_i)/H^2(Q_i). \tag{2.4}$$

Since $H^2(\tilde{Y}) = \phi^* H^2(Y) \oplus \bigoplus_{i=1}^k \langle [E_i] \rangle$ and $[E_i]|_{E_i} = -(\ell_i + \ell'_i)$, the second component $\bigoplus_{i=1}^k \langle [E_i] \rangle$ lies in $\ker(\bar{\delta}_2)$ and $\bar{\delta}_2$ factors through

$$\phi^* H^2(Y) \to \bigoplus_{i=1}^{k} H^2(E_i)/H^2(Q_i) \cong \bigoplus_{i=1}^{k} \langle \ell_i - \ell'_i \rangle \tag{2.5}$$

(as $\mathbb{Q}$-spaces). Notice that the quotient is isomorphic to $\bigoplus_{i=1}^k \langle \ell'_i \rangle$ integrally.

By reordering we may assume that $\phi_* \ell_i = [C_i]$ and $\phi^*[C_i] = \ell_i - \ell'_i$ (c.f. [16]). The dual of (2.5) then coincides with the fundamental class map

$$\vartheta : \bigoplus_{i=1}^{k} \langle [C_i] \rangle \longrightarrow H_2(Y).$$

In general for a $\mathbb{Q}$-linear map $\vartheta : P \to Z$, we have

$$\operatorname{im} \vartheta^* \cong (P/\ker \vartheta)^* \cong (\operatorname{im} \vartheta)^*.$$

Thus

$$\dim_{\mathbb{Q}} \operatorname{cok}(\delta_2) + \dim_{\mathbb{Q}} \operatorname{im}(\vartheta) = k. \tag{2.6}$$

We will see in Corollary 2.5 that $\dim \operatorname{cok} \delta = \mu$ and $\dim \operatorname{im} \vartheta = \rho$. This gives the Hodge theoretic meaning of $\mu + \rho = k$ in Lemma 1.7. Further elaboration of this theme will follow in Theorem 2.9.

2.2.3. On $\mathcal{Y}_0$, the computation is similar and a lot easier. The weight 3 piece can be computed by the map

$$H^3(Y^{[0]}) = H^3(\tilde{Y}) \longrightarrow H^3(Y^{[1]}) = 0;$$

the weight 2 piece is similarly computed by the map

$$H^2(Y^{[0]}) = H^2(\tilde{Y}) \oplus \bigoplus_{i=1}^{k} H^2(\tilde{E}_i) \xrightarrow{\delta'_2} H^2(Y^{[1]}) = \bigoplus_{i=1}^{k} H^2(E_i).$$

Let $h = \pi^*(\mathrm{pt})$ and $\xi = [E]$ for

$$\pi : \tilde{E} \to \mathbb{P}^1.$$

Then $h|_E = \ell'$ and $\xi|_E = \ell + \ell'$. In particular the restriction map $H^2(\tilde{E}) \to H^2(E)$ is an isomorphism and hence δ'_2 is surjective. The computation of pieces from weights 1 and 0 is the same as for $\mathcal{X}_0$. We have therefore the following lemma.

Lemma 2.2.

$$H^3(\mathcal{Y}_0) \cong H^3(Y^{[0]}) \cong H^3(\tilde{Y}),$$
$$H^2(\mathcal{Y}_0) \cong \ker(\delta_2') \cong H^2(\tilde{Y}),$$
$$H^1(\mathcal{Y}_0) \cong H^1(\tilde{Y}) \cong H^1(Y) \cong H^1(X),$$
$$H^0(\mathcal{Y}_0) \cong H^0(\tilde{Y}) \cong H^0(Y) \cong H^0(X).$$

2.2.4. Slightly abusing the notation, we denote by N the monodromy operator for both $\mathcal{X}$ and $\mathcal{Y}$ families. N induces the weight filtrations on Schmid's limiting Hodge structures on $H^*(X)$ and $H^*(Y)$.

Lemma 2.3. *We have the following exact sequences (of MHS) for H^2 and H^3 of $\mathcal{X}_0$ and $\mathcal{Y}_0$:*

$$0 \to H^3(\mathcal{X}_0) \to H^3(X) \xrightarrow{N} H^3(X) \to H_3(\mathcal{X}_0) \to 0,$$
$$0 \to H^0(X) \to H_6(\mathcal{X}_0) \to H^2(\mathcal{X}_0) \to H^2(X) \xrightarrow{N} 0,$$
$$0 \to H^3(\mathcal{Y}_0) \to H^3(Y) \xrightarrow{N} 0,$$
$$0 \to H^0(Y) \to H_6(\mathcal{Y}_0) \to H^2(\mathcal{Y}_0) \to H^2(Y) \xrightarrow{N} 0,$$

Proof. These follow from the Clemens–Schmid exact sequence, which is compatible with the MHS. Note that the monodromy is trivial for $\mathcal{Y} \to \Delta$ since the punctured family is trivial. By Lemma 2.1, we know that $H^2(\mathcal{X}_0)$ is pure of weight 2. Hence N on $H^2(X)$ is also trivial and the Hodge structure does not degenerate. □

Remark 2.4. Strictly speaking there are other terms in the first sequence, namely $H^1(X) \to H_5(\mathcal{X}_0)$ to the left end and $H^5(\mathcal{X}_0) \to H^5(X)$ to the right end. It can be ignored since they induce isomorphisms, as can be checked using MHS on $H_5(\mathcal{X}_0)$. Similar comments apply to the third sequence for $H^3(Y)$. All these vanish if we impose the regularity condition $h^1(\mathcal{O}) = 0$.

Corollary 2.5. (i) $\rho = \dim \operatorname{im}(\vartheta)$ *and* $\mu = \dim \operatorname{cok}(\delta_2)$.
(ii) $H^3(Y) \cong H^3(\mathcal{Y}_0) \cong H^3(Y^{[0]}) \cong H^3(\tilde{Y}) \cong \operatorname{Gr}_3^W H^3(X)$.
(iii) *Denote by K the kernel of the monodromy operator*

$$K := \ker(N : H^3(X) \to H^3(X)).$$

We have $H^3(\mathcal{X}_0) \cong K$. More precisely,

$$\operatorname{Gr}_3^W(H^3(\mathcal{X}_0)) \cong H^3(Y), \qquad \operatorname{Gr}_2^W(H^3(\mathcal{X}_0)) \cong \operatorname{cok}(\delta_2).$$

Proof. By Lemma 2.1, $h^2(\mathcal{X}_0) = \dim \ker(\delta_2)$. It follows from the second and the fourth exact sequences in Lemma 2.3 that

$$h^2(X) = \dim \ker(\delta_2) + 1 - (k+1).$$

Rewrite (2.3) as

$$0 \to \ker(\delta_2) \to H^2(X^{[0]}) \xrightarrow{\delta} H^2(X^{[1]}) \to \operatorname{cok}(\delta_2) \to 0, \tag{2.7}$$

which implies

$$\dim \ker(\delta_2) + 2k = \dim \operatorname{cok}(\delta_2) + 2k + h^2(Y).$$

Combining these two equations with (2.6), we have

$$\rho = h^2(Y) - h^2(X) = k - \dim \operatorname{cok}(\delta_2) = \dim \operatorname{im}(\vartheta).$$

This proves the first equation for ρ in (i).

Combining the first equation in Lemma 2.2 and the third exact sequence in Lemma 2.3, we have

$$H^3(Y) \cong H^3(\mathcal{Y}_0) \cong H^3(\tilde{Y}). \tag{2.8}$$

(This can also be seen from the geometry of blowing up.) This shows (ii) except the last equality.

By Lemmas 2.3 and 2.1,

$$K \cong H^3(\mathcal{X}_0) \cong H^3(\tilde{Y}) \oplus \operatorname{cok}(\delta_2) \cong H^3(Y) \oplus \operatorname{cok}(\delta_2),$$

where the last equality follows from (2.8). This proves (iii).

For the remaining parts of (i) and (ii): From the non-trivial terms of the limiting Hodge diamond, where $H^n := H^n(X)$ and

$$H^{p,q}_\infty H^n = F^p_\infty \operatorname{Gr}^W_{p+q} H^n,$$

we have

$$\begin{array}{ccccccc} & & & H^{2,2}_\infty H^3 & & & \\ & & \diagup & \big\downarrow \sim N & & & \\ H^{3,0}_\infty H^3 & H^{2,1}_\infty H^3 & & & & H^{1,2}_\infty H^3 & H^{0,3}_\infty H^3 \\ & & & & \diagup & & \\ & & & H^{1,1}_\infty H^3, & & & \end{array} \tag{2.9}$$

where $H^{3,0}_\infty H^3$ does not degenerate due to a result in [38] (which holds for more general degenerations with canonical singularities, and first proved in [37] for the Calabi–Yau case). We conclude that $H^{1,1}_\infty H^3 \cong \operatorname{cok}(\delta_2)$ and $\operatorname{Gr}^W_3 H^3(X) \cong H^3(Y)$. Thus

$$\mu = h^{2,2}_\infty H^3 = h^{1,1}_\infty H^3 = \dim \operatorname{cok}(\delta_2).$$

□

2.2.5. We denote the *vanishing cycle space* V as the $\mathbb{Q}$-vector space generated by vanishing 3-cycles. We first define the abelian group $V_\mathbb{Z}$ from

$$0 \to V_\mathbb{Z} \to H_3(X,\mathbb{Z}) \to H_3(\bar{X},\mathbb{Z}) \to 0, \tag{2.10}$$

and $V := V_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Q}$. We note that the exactness on the right holds for any 3-fold isolated singularities.

We will give a further geometric characterization of the defect invariant μ in terms of V.

Lemma 2.6. (i) $H^3(\bar{X}) \cong K \cong H^3(\mathfrak{X}_0)$.
(ii) $V^* \cong H^{2,2}_\infty H^3$ *and* $V \cong H^{1,1}_\infty H^3 = \operatorname{cok}(\delta_2)$.

Proof. Dualizing (2.10) over $\mathbb{Q}$, we have

$$0 \to H^3(\bar{X}) \to H^3(X) \to V^* \to 0.$$

The invariant cycle theorem in [1] implies that

$$H^3(\bar{X}) \cong \ker N = K \cong H^3(\mathcal{X}_0).$$

This proves (i).

Hence we have the canonical isomorphism

$$V^* \cong H^{2,2}_\infty H^3 = F^2_\infty G^W_4 H^3(X).$$

Moreover, the non-degeneracy of the pairing $(N\alpha, \beta)$ on $G^W_4 H^3(X)$ implies that

$$H^{1,1}_\infty H^3 = N H^{2,2}_\infty H^3 \cong (H^{2,2}_\infty H^3)^* \cong V^{**}_{\mathbb{C}} \cong V_{\mathbb{C}}.$$

This proves (ii). □

Remark 2.7. We must be careful in dealing with this isomorphism $H^{1,1}_\infty H^3 \cong V$. The vanishing cycle space V is defined over $\mathbb{Z}$ while $H^{1,1}_\infty H^3$ is intrinsically defined only as a complex vector space. In identifying V with $H^{1,1}_\infty H^3$, we used two different duality: $\operatorname{Hom}(\cdot, \mathbb{Q})$, which brings it to the dual space, and the duality under a bilinear pairing (Poincaré pairing), which stays in the same vector space.

Remark 2.8 (On threefold extremal transitions). Most results in Section 2.2 works for more general geometric contexts. The mixed Hodge diamond (2.9) holds for any 3-folds degenerations with at most canonical singularities [38]. The identification of vanishing cycle space V via (2.10) works for 3–folds with only isolated singularities, hence Lemma 2.6 works for any 3-fold degenerations with isolated canonical singularities.

Later on we will impose the Calabi–Yau condition on all the 3-folds involved. If $X \nearrow Y$ is a terminal transition of Calabi–Yau 3-folds, i.e., $\mathfrak{X}_0 = \bar{X}$ has at most (isolated Gorenstein) terminal singularities, then $\bar{X}$ has unobstructed deformations [26]. Moreover, the small resolution $Y \to \bar{X}$ induces an embedding $\operatorname{Def}(Y) \hookrightarrow \operatorname{Def}(\bar{X})$ which identifies the limiting/ordinary pure Hodge structures $\operatorname{Gr}^W_3 H^3(X) \cong H^3(Y)$ as in Corollary 2.5 (iii).

For conifold transitions all these can be described in explicit terms and more precise structure will be formulated.

2.3. **The basic exact sequence.** We may combine the four Clemens–Schmid exact sequences into one short exact sequence, which we call the *basic exact sequence*, to give the Hodge-theoretic realization of the equality "$\rho + \mu = k$" in Lemma 1.7.

Let $A = (a_{ij}) \in M_{k\times\mu}(\mathbb{Z})$ be the relation matrix for C_i's, i.e.,

$$\sum_{i=1}^{k} a_{ij}[C_i] = 0, \qquad j = 1, \ldots, \mu.$$

Similarly, let $B = (b_{ij}) \in M_{k\times\rho}(\mathbb{Z})$ be the relation matrix for S_i's:

$$\sum_{i=1}^{k} b_{ij}[S_i] = 0, \qquad j = 1, \ldots, \rho.$$

Theorem 2.9 (Basic exact sequence)**.** *The group of real 2-cycles generated by exceptional curves C_i (vanishing S^2 cycles) on Y and the group of 3-cycles generated by $[S_i]$ (vanishing S^3 cycles) on X are linked by the following weight 2 exact sequence*

$$0 \to H^2(Y)/H^2(X) \xrightarrow{B} \bigoplus_{i=1}^{k} H^2(E_i)/H^2(Q_i) \xrightarrow{A^t} V \to 0.$$

In particular $B = \ker A^t$ and $A = \ker B^t$.

Proof. To see this, we use the sequence in (2.7). From the discussions in Section 2.2.2, we know that $\operatorname{cok}(\delta_2) = \operatorname{cok}(\bar{\delta}_2)$ and (2.7) can be replaced by

$$0 \to H^2(\tilde{Y})/(\ker \bar{\delta}) \xrightarrow{D} \bigoplus_{i=1}^{k} H^2(E_i)/H^2(Q_i) \xrightarrow{C} \operatorname{cok}(\delta_2) \to 0. \tag{2.11}$$

By Lemma 2.6 (ii), we have $\operatorname{cok}(\delta_2) \cong V$. To prove the theorem, we need to show that

$$H^2(\tilde{Y})/\ker \bar{\delta} \cong H^2(Y)/H^2(X),$$

and $D = B$, $C = A^t$.

Let us start with making sense of the quotient $H^2(Y)/H^2(X)$. Again by the version of invariant cycle theorem in [1], we have $H^2(X) = H^2(\bar{X})$. By the blow-up description in Section 1.1, $H^2(\bar{X})$ injects to $H^2(Y)$ by pullback. This defines the embedding

$$\iota : H^2(X) \hookrightarrow H^2(Y) \tag{2.12}$$

and the quotient $H^2(Y)/H^2(X)$.

Recast the relation matrix A of the rational curves C_i in the following form

$$0 \to \mathbb{Q}^{\mu} \xrightarrow{A} \mathbb{Q}^k \cong \bigoplus_{i=1}^{k} \langle [C_i] \rangle \xrightarrow{S} \operatorname{im}(\vartheta) \to 0$$

where $S = \operatorname{cok}(A) \in M_{\rho\times k}$ is the matrix for ϑ, and $\operatorname{im}(\vartheta)$ has rank ρ. The dual sequence reads

(2.13)

$$0 \to (\operatorname{im}\vartheta)^* \cong (\mathbb{Q}^{\rho})^* \xrightarrow{S^t} (\mathbb{Q}^k)^* \cong \bigoplus_{i=1}^{k} H^2(E_i)/H^2(Q_i) \xrightarrow{A^t} (\mathbb{Q}^{\mu})^* \to 0.$$

Compare (2.13) with (2.11), we see that $(\mathbb{Q}^\mu)^* \cong V$. From the discussion in Section 2.2.2, we have $(\operatorname{im}\vartheta)^* = H^2(Y)/H^2(X)$.

We want to reinterpret the map $A^t : (\mathbb{Q}^k)^* \to V$ in (2.13). This is a presentation of V by k generators, denoted by σ_i, and the relation matrix of which is given by S^t. If we show that σ_i can be identified with the vanishing sphere S_i, then $(\mathbb{Q}^\mu)^* \cong V$ and $B = S^t = \ker A^t$ is the relation matrix for S_i's.

Consider the following topological construction. For any non-trivial integral relation $\sum_{i=1}^k a_i[C_i] = 0$, there is a 3-chain θ in Y with

$$\partial\theta = \sum_{i=1}^k a_i C_i.$$

Under $\psi : Y \to \bar{X}$, C_i collapses to the node p_i. Hence it creates a 3-cycle $\bar{\theta} := \psi_*\theta \in H_3(\bar{X}, \mathbb{Z})$, which deforms (lifts) to $\gamma \in H_3(X, \mathbb{Z})$ in nearby fibers. Using the intersection pairing on $H_3(X, \mathbb{Z})$, γ then defines an element $\mathrm{PD}(\gamma)$ in $H^3(X, \mathbb{Z})$. Under the restriction to the vanishing cycle space V, we get $\mathrm{PD}(\gamma) \in V^*$.

It remains to show that

$$(\gamma.S_i) = a_i.$$

Let U_i be a small tubular neighborhood of S_i and $\tilde{U}_i$ be the corresponding tubular neighborhood of C_i, then by Corollary 1.5,

$$\partial U_i \cong \partial(S_i^3 \times D^3) \cong S^3 \times S^2 \cong \partial(D^4 \times C_i) \cong \partial\tilde{U}_i.$$

Now $\theta_i := \theta \cap \tilde{U}_i$ gives a homotopy between $a_i[C_i]$ (in the center of $\tilde{U}_i$) and $a_i[S^2]$ (on $\partial\tilde{U}_i$). Denote by $\iota : \partial U_i \hookrightarrow X$ and $\tilde{\iota} : \partial\tilde{U}_i \hookrightarrow Y$. Then

$$\begin{aligned}(\gamma.S_i)^X &= (\gamma.\iota_*[S^3])^X = (\iota^*\gamma.[S^3])^{\partial U_i} = (\tilde{\iota}^*\gamma.[S^3])^{\partial\tilde{U}_i} \\ &= (a_i[S^2], [S^3])^{S^3\times S^2} = a_i.\end{aligned}$$

The proof is complete. □

Remark 2.10. As a byproduct, notice that there are precisely $k - \rho = \mu = \dim V^*$ independent relations, hence we also see directly that $(a_i) \mapsto \mathrm{PD}(\gamma)$ establishes a group isomorphism from curve relations among C_i's to V^*.

Convention. We would like to choose a preferred basis of the vanishing cocycles V^* as well as a basis of divisors dual to the space of extremal curves. These notations will fixed henceforth and will be used in later sections.

During the course of the proof of Theorem 2.9 (c.f. Remark 2.10) we establish the correspondence for each column vector $A^j = (a_{1j}, \cdots, a_{kj})^t$ with the element $\mathrm{PD}(\gamma_j) \in V^*$, $1 \le j \le \mu$, characterized by

$$a_{ij} = (\gamma_j.S_i).$$

The subspace of $H_3(X)$ spanned by these γ_j's will be denoted by V'.

Dually, we denote by $T_1, \cdots, T_\rho \in H^2(Y)$ those divisors which form an integral basis of the lattice in $H^2(Y)$ dual (orthogonal) to $H_2(X) \subset H_2(Y)$.

In particular they form an integral basis of $H^2(Y)/H^2(X)$. Notice that we may choose T_l's such that T_l corresponds to the l-th column vector of the matrix B via

$$b_{il} = (C_i.T_l).$$

Such a choice is consistent with the basic exact sequence since

$$(A^t B)_{jl} = \sum_{i=1}^{k} a^t_{ji} b_{il} = \sum_{i=1}^{k} a_{ij}(C_i.T_l) = \Big(\sum a_{ij}[C_i]\Big).T_l = 0$$

for all j, l. We may also assume that the first $\rho \times \rho$ minor of B has full rank.

3. Gromov–Witten theory and Dubrovin connections

3.1. **Consequences of the degeneration formula for threefolds.** Gromov–Witten theory on X can be related to that on Y by the degeneration formula through the two semistable degenerations introduced in Section 2.1.

In the previous section, we have seen that the monodromy actions are trivial on $H(X)$ except $H^3(X)$ for which we have

$$H^3_{inv}(X) = K \cong H^3(Y) \oplus H^{1,1}_\infty H^3(X) \cong H^3(Y) \oplus V.$$

There we implicitly have a linear map

$$\iota : H^j_{inv}(X) \to H^j(Y) \tag{3.1}$$

as follows. For $j = 3$, it is the projection

$$H^3_{inv}(X) \cong H^3(Y) \oplus V \to H^3(Y).$$

For $j = 2$, it is the embedding defined in (2.12) and $j = 4$ case is the same as (dual to) $j = 2$ case. For $j = 0, 1, 5, 6$, ι is an isomorphism.

The following is a refinement of a result of Li–Ruan [22]. (See also [23].)

Proposition 3.1. *Let $X \nearrow Y$ be a projective conifold transition. Given*

$$\vec{a} \in (H^{\ge 2}_{inv}(X)/V)^{\oplus n}$$

and a curve class $\beta \in NE(X) \setminus \{0\}$, we have

$$\langle \vec{a} \rangle^X_{g,n,\beta} = \sum_{\psi_*(\gamma)=\beta} \langle \iota(\vec{a}) \rangle^Y_{g,n,\gamma}. \tag{3.2}$$

If some component of $\vec{a}$ lies in H^0, then both sides vanish. Furthermore, the RHS is a finite *sum.*

Proof. (3.2) has been proved in [22, 23] under slightly stronger assumptions. We review its proof with slight refinements as it will be useful in Section 5.

A cohomology class $a \in H^{>2}_{inv}(X)/V$ can always find an admissible lift to

$$(a_i)_{i=0}^k \in H(\tilde{Y}) \oplus \bigoplus_{i=1}^{k} H(Q_i)$$

such that $a_i = 0$ for all $i \neq 0$. This is the lifting of the cohomology class we will use in the degeneration arguments below.

We apply J. Li's algebraic version of degeneration formula [21, 23] to the complex degeneration (2.1) $X \rightsquigarrow \tilde{Y} \cup_E Q$, where $Q = \coprod Q_i$ is a disjoint union of quadrics Q_i's and $E := \sum_{i=1}^k E_i$. One has $K_{\tilde{Y}} = \tilde{\psi}^* K_{\bar{X}} + E$. The topological data (g, n, β) lifts to two admissible triples Γ_1 on $(\tilde{Y}, E)$ and Γ_2 on (Q, E) such that Γ_1 has curve class $\tilde{\gamma} \in NE(\tilde{Y})$, contact order $\mu = (\tilde{\gamma}.E)$, and number of contact points ρ. Then

$$(\tilde{\gamma}.c_1(\tilde{Y})) = (\tilde{\psi}_* \tilde{\gamma}.c_1(\bar{X})) - (\tilde{\gamma}.E) = (\beta.c_1(X)) - \mu.$$

The virtual dimension (without marked points) is given by

$$\begin{aligned} d_{\Gamma_1} &= (\tilde{\gamma}.c_1(\tilde{Y})) + (\dim X - 3)(1-g) + \rho - \mu \\ &= d_\beta + \rho - 2\mu. \end{aligned}$$

Since we chose the lifting $(\vec{a}_i)_{i=0}^k$ of $\vec{a}$ to have $\vec{a}_i = 0$ for all $i \neq 0$, all insertions contribute to $\tilde{Y}$. If $\rho \neq 0$ then $\rho - 2\mu < 0$. This leads to vanishing relative GW invariant on $(\tilde{Y}, E)$. Therefore, ρ must be zero. To summarize, we get

$$\langle \vec{a} \rangle^X_{g,n,\beta} = \sum_{\tilde{\psi}_*(\tilde{\gamma}) = \beta} \langle \vec{a}_0 \mid \emptyset \rangle^{(\tilde{Y},E)}_{g,n,\tilde{\gamma}}, \tag{3.3}$$

such that

$$\tilde{\psi}_* \tilde{\gamma} = \beta, \qquad \tilde{\gamma}.E = 0, \qquad \tilde{\gamma}_Q = 0. \tag{3.4}$$

We note that this equation also holds for a_i a divisor by the divisor axiom.

We use a similar argument to compute $\langle \vec{b} \rangle^Y_{g,n,\gamma}$ via the Kähler degeneration (2.2) $Y \rightsquigarrow \tilde{Y} \cup \tilde{E}$, where $\tilde{E}$ is a disjoint union of $\tilde{E}_i$ (cf. [16, Theorem 4.10]). By the divisor equation we may assume that $\deg b_j \geq 3$ for all $j = 1, \ldots, n$. We still choose the lifting $(\vec{b})_{i=0}^k$ of $\vec{b}$ such that $\vec{b}_i = 0$ for all $i \neq 0$. In the lifting γ_1 on $\tilde{Y}$ and γ_2 on $\pi : \tilde{E} = \coprod_i \tilde{E}_i \to \coprod_i C_i$, we must have $\gamma = \phi_* \gamma_1 + \pi_* \gamma_2$. The contact order is given by $\mu = (\gamma_1.E)$ which has the property that $\mu = 0$ if and only if $\gamma_1 = \phi^* \gamma$ (and hence $\gamma_2 = 0$). If $\rho \neq 0$ we still get

$$d_{\Gamma_1} = d_\gamma + \rho - 2\mu < d_\gamma$$

and the invariant is thus zero. This proves that

$$\langle \vec{b} \rangle^Y_{g,n,\gamma} = \langle \phi^* \vec{b} \mid \emptyset \rangle^{(\tilde{Y},E)}_{g,n,\phi^*\gamma}, \tag{3.5}$$

such that

$$\phi_* \tilde{\gamma} = \gamma, \qquad \tilde{\gamma}.E = 0, \qquad \tilde{\gamma}_{\tilde{E}} = 0. \tag{3.6}$$

To combine these two degeneration formulas together, we notice that in the Kähler degeneration, $\tilde{\gamma} \in NE(\tilde{Y})$ can have contact order $\mu = (\tilde{\gamma}.E) = 0$ if and only if $\tilde{\gamma} = \phi^* \gamma$ for some $\gamma \in NE(Y)$ (indeed for $\gamma = \phi_* \tilde{\gamma}$). Choose $\vec{b} = \iota(\vec{a})$ and the formula in the proposition follows.

The vanishing statement (of H^0 insertion) follows from the fundamental class axiom.

Now we proceed to prove the finiteness of the sum. (This part is not stated in [22].) For $\phi : \tilde{Y} \to Y$ being the blow-up along C_i's, the curve class $\gamma \in NE(Y)$ contributes a non-trivial invariant in the sum only if $\phi^* \gamma$ is effective on $\tilde{Y}$. By combining (2.5), (3.3) and (3.5), the effectivity of $\phi^* \gamma$ forces the sum to be finite. Equivalently, the condition that $\phi^* \gamma$ is effective is equivalent to that γ is $\mathcal{F}$-effective under the flop

$$\begin{array}{ccc} Y & \dashrightarrow & Y' \\ & \searrow \quad \swarrow & \\ & \bar{X} & \end{array}$$

(i.e. effective in Y and in Y' under the natural correspondence [16]). Recall that under the flop the flopping curve class in Y is mapped to the negative flopping curve in Y'. Therefore, the sum is finite. □

Remark 3.2. (i) The phenomena, including finiteness of the sum, were observed in [12] for Calabi–Yau hypersurfaces in weighted projective spaces from the numerical data obtained from the corresponding B model generating function and mirror symmetry.

(ii) For general 3-folds extremal transitions worse than conifolds, the double point degeneration formula does not apply directly. In general, the relative GW invariants will enter the degeneration formula in an essential way and can not be reduced to the absolute GW theory in a simple explicit way as above. For example, in higher dimensions none of the complicated features in the degeneration formula can be avoided.

Corollary 3.3. *Gromov–Witten theory on even cohomology $GW^{ev}(X)$ (of all genera) can be considered as a proper sub-theory of $GW^{ev}(Y)$.*

In particular, the big quantum cohomology ring is functorial with respect to $\iota : H^{ev}(X) \to H^{ev}(Y)$ in (3.1).

Proof. We first note that ι is an injection on H^{ev}. Proposition 3.1 then implies that all Gromov–Witten invariants of X with even classes can be recovered from invariants of Y. The only exception, H^0, can be treated by the fundamental class axiom. Therefore, in this sense that $GW^{ev}(X)$ is a sub-theory of $GW^{ev}(Y)$.

In genus zero, however, there is a more precise sense of being a sub-theory via functoriality. Observe that the degeneration formula also holds for $\beta = 0$. For $g = 0$, this leads to the equality of classical triple product on $H_{inv}(X)$ under ι:

$$(a, b, c)^X = (\iota(a), \iota(b), \iota(c))^Y.$$

Since the Poincaré pairing on $H^{ev}(X)$ is also preserved under ι, we see that the classical ring structure on $H^{ev}(X)$ are naturally embedded in $H^{ev}(Y)$.

To see the functoriality of the big quantum ring with respect to ι, we note that $(\iota(a).C_i) = 0$ for any $a \in H^{ev}(X)$ and for any extremal curve C_i in Y. Furthermore, for the invariants associated to the extremal rays the insertions must involve only divisors by the virtual dimension count. Hence in the level of generating functions with *at least one insertion* we also have

$$\sum_{\beta \in NE(X)} \langle \vec{a} \rangle_\beta^X q^\beta = \sum_{\gamma \in NE(Y)} \langle \iota(\vec{a}) \rangle_\gamma^Y q^{\psi_*(\gamma)}.$$

Note that the case of H^0 is not covered in Proposition 3.1, but can be treated by the fundamental class axiom as above. □

Remark 3.4. It is clear that the argument and conclusion hold even if some insertions lie in $H^3_{inv}(X)/V \cong H^3(Y)$ by Proposition 3.1.

The full GW theory is built on the full cohomology *superspace* $H = H^{ev} \oplus H^{odd}$. However, the odd part is not as well-studied in the literature as the even one. In some special cases the difficulty does not occur for elementary reasons.

Lemma 3.5. *Let X be a smooth minimal 3-fold (e.g., Calabi–Yau threefold) with $H^1(X) = 0$. The non-trivial primary GW invariants are all supported on $H^2(X)$.*

More generally the conclusion holds for any curve class $\beta \in NE(X)$ with $c_1(X).\beta \le 0$ for any 3-fold X with $H^1(X) = 0$.

Proof. For n-point invariants, the virtual dimension of $\overline{M}_{g,n}(X, \beta)$ is given by

$$c_1(X).\beta + (\dim X - 3)(1 - g) + n \le n.$$

Since the appearance of fundamental class in the insertions leads to trivial invariants, we must have the algebraic degree $\deg a_i \ge 1$ for all insertions a_i, $i = 1, \ldots, n$. Hence in fact we must have $\deg a_i = 1$ for all i and $c_1(X).\beta = 0$. □

Remark 3.6. By the divisor axiom, the primary GW theory for smooth minimal 3-folds is then completely reduced to the case without any insertions.

3.2. **The even and extremal quantum cohomology.** From now on, we restrict to genus zero theory.

3.2.1. For simplicity we restrict our discussions on insertions $s = \sum_\epsilon s^\epsilon \bar{T}_\epsilon \in H^2(X)$ where $\bar{T}_\epsilon$'s form a basis of $H^2(X)$. Then the genus zero GW prepotential is given by

$$F_0^X(s) = \sum_{n=0}^\infty \sum_{\beta \in NE(X)} \langle s^n \rangle_{0,n,\beta} \frac{q^\beta}{n!} = \frac{s^3}{3!} + \sum_{\beta \neq 0} n_\beta^X q^\beta e^{(\beta.s)}, \tag{3.7}$$

where $n_\beta^X = \langle\rangle_{0,0,\beta}^X$, with formal variables q^β's. It can be considered as a function in the "Kähler moduli" via identification

$$q^\beta = \exp 2\pi\sqrt{-1}(\beta.\omega),$$

where

$$\omega = B + \sqrt{-1}H \in \mathcal{K}_{\mathbb{C}}^X := H^2(X) + \sqrt{-1}\mathcal{K}^X,$$

the complexified Kähler cone of X.

$F_0^X(s)$ *almost* gives the small quantum cohomology of X. (By Lemma 3.5, this is the same as big quantum cohomology if X is a Calabi–Yau threefold.) In order to have the full small quantum cohomology ring, we will need to consider $s \in H^{ev}(X)$ in the first term $s^3/(3!)$, which will be called classical, topological or cubic terms. Namely, in terms of dual basis notations,

$$s = s^0 \bar{T}_0 + \sum_\epsilon s^\epsilon \bar{T}_\epsilon + \sum_\zeta s_\zeta \bar{T}^\zeta + s_0 \bar{T}^0 \in H^0 \oplus H^2 \oplus H^4 \oplus \oplus H^6$$

and

$$\frac{s^3}{3!} = \frac{1}{3!}\Big(\sum_\epsilon s^\epsilon \bar{T}_\epsilon\Big)^3 + \frac{1}{2}(s^0)^2 s_0 + s^0 \sum_\epsilon s^\epsilon s_\epsilon.$$

For simplicity of notation, and without loss of generality, we treat the divisor variables first and bring back the other two topological terms when we need to write down the complete Dubrovin connection.

Remark 3.7. The reader might consider all the following discussions are for the big quantum cohomology of a Calabi–Yau threefolds, since that is the case we will be primarily concerned with in the later sections.

Remark 3.8. In practice, the variables s and ω encode basically equivalent information: By divisor axiom, q^β always appears in the form $q^\beta \exp(\beta, s)$. If there is no convergence issue then it makes no essential difference to drop out the Novikov variables by setting $q^\beta \equiv 1$ for all β.

3.2.2. Similarly we have $F_0^Y(t)$ on $H^2(Y) \times \mathcal{K}_{\mathbb{C}}^Y$. Here we use the variable

$$t = s + u \in H^2(Y) = \iota(H^2(X)) \oplus \bigoplus_{l=1}^{\rho} \langle T_l \rangle.$$

Namely we identify s with $\iota(s)$ in $H^2(Y)$ and write $u = \sum_{l=1}^\rho u^l T_l$. F_0^Y can be analytically continued across those boundary faces of $\mathcal{K}_{\mathbb{C}}^Y$ which corresponds to flopping contractions. In the case of conifold transitions $Y \searrow X$, the boundary face is precisely $\mathcal{K}_{\mathbb{C}}^X \subset \bar{\mathcal{K}}_{\mathbb{C}}^Y$.

Convention. The following convention of indices on $H^{ev}(Y)$ will be used throughout the rest of this section:

- Lowercase Greek alphabets for indices from the subspace $\iota(H^{ev}(X))$;
- lowercase Roman alphabets for indices from the subspace spanned by the divisors T_l's and exceptional curves C_i's;
- uppercase Roman alphabets for variables from the total space $H^{ev}(Y)$.

For $C \cong \mathbb{P}^1$ with twisted bundle $N = \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$, the extremal function is given by the well-known multiple cover formula

$$E_0^C(t) = \sum_{d \in \mathbb{N}} n_d^N q^{d[C]} e^{d(C.t)} = \sum_{d \in \mathbb{N}} \frac{1}{d^3} q^{d[C]} e^{d(C.t)}.$$

We also consider the total (global) extremal function

$$E_0^Y(t) := \frac{t^3}{3!} + \sum_{i=1}^{k} E_0^{C_i}(t) = E_0^Y(u) + \frac{1}{3!}(t^3 - u^3),$$

where we notice that $E_0^{C_i}(t) = E_0^{C_i}(u)$ depends only on u.

Then the degeneration formula is equivalent to the following restriction

$$F_0^X(s) - \frac{s^3}{3!} = \Big(F_0^Y(s+u) - \frac{(s+u)^3}{3!} - E_0^Y(u) + \frac{u^3}{3!}\Big)\Big|_{q^{\gamma} \mapsto q^{\psi_*(\gamma)}},$$

or equivalently the restriction of Kähler moduli to the boundary face $\mathcal{K}_{\mathbb{C}}^X$. Notice that the Novikov variables $q^{[C_i]}$'s are subject to the relations: For $\sum_{i=1}^k a_{ij}[C_i] = 0$ in $NE(Y)$ with $A = (a_{ij}) \in M_{k\times\mu}(\mathbb{Z})$ being the relation matrix, we define

$$\mathbf{r}_j(q) := \prod_{a_{ij}>0} q^{a_{ij}[C_i]} - \prod_{a_{ij}<0} q^{-a_{ij}[C_i]}$$

and force the relation $\mathbf{r}_j(q) = 0$ for $1 \le j \le \mu$ since they vanish trivially on the Kähler moduli.

Summarizing the above discussion, we have

Lemma 3.9.

$$F_0^Y(s+u) = \left[F_0^X(s) + E_0^Y(u) + \frac{1}{3!}((s+u)^3 - s^3 - u^3)\right]_{\mathbf{r}_j(q)=0,\,1\le j\le\mu}.$$

Convention. For simplicity of notation, we restrict the Novikov variables implicitly and drop it from the notation henceforth.

A complete splitting of variables of the pre-potential function F_0^Y would imply that the big quantum cohomology $QH^{ev}(Y)$ decomposes into two blocks. One piece is identified with $QH^{ev}(X)$, and another piece with contributions from the extremal rays. However, the classical cup product terms enter into the formula and destroy the complete splitting. Thus the two pieces are not completely independent.

3.2.3. The structural coefficients for $QH^{ev}(Y)$ are $C_{PQR} = \partial^3_{PQR} F_0^Y$. We will determine them according to the above splitting.

For $F_0^X(s)$, the structural coefficients of quantum product are given by

$$C_{\epsilon\zeta\iota} := \partial^3_{\epsilon\zeta\iota} F_0^X(s) = (\bar{T}_\epsilon.\bar{T}_\zeta.\bar{T}_\iota) + \sum_{\beta\neq 0} (\beta.\bar{T}_\epsilon)(\beta.\bar{T}_\zeta)(\beta.\bar{T}_\iota)\, n_\beta^X\, q^\beta e^{(\beta.s)}.$$

Recall that $B = (b_{ip})$ with $b_{ip} = (C_i.T_p)$ is the relation matrix for the vanishing 3-spheres. For $E_0^Y(u)$, the triple derivatives are

$$\begin{aligned} C_{lmn} &:= \partial^3_{lmn} E_0^Y(u) \\ &= (T_l.T_m.T_n) + \sum_{i=1}^{k} \sum_{d\in\mathbb{N}} (C_i.T_l)(C_i.T_m)(C_i.T_n)\, q^{d[C_i]} e^{d(C_i.u)} \\ &= (T_l.T_m.T_n) + \sum_{i=1}^{k} b_{il} b_{im} b_{in} \mathbf{f}(q^{[C_i]} \exp \textstyle\sum_{p=1}^{\rho} b_{ip} u^p). \end{aligned} \tag{3.8}$$

Here

$$\mathbf{f}(q) = \sum_{d\in\mathbb{N}} q^d = \frac{q}{1-q} = -1 + \frac{-1}{q-1} \tag{3.9}$$

is the fundamental rational function with a simple pole at $q = 1$ with residue -1. ($\mathbf{f}(q)$ plays an important role in our study of GW invariants associated to a flopping contraction in [16] and subsequent works.)

However, due to the existence of possible cross terms, C_{lmn}'s do not satisfy the WDVV equations. Indeed, the remaining cross terms are

$$\theta(s+u) := \frac{1}{2}(s^2u + su^2) = \frac{1}{2}su^2.$$

The first term $s^2u = 0$ since T_l's are chosen to be orthogonal to $NE(X)$. Then the only non-trivial mixed triple derivatives are constants (cup product)

$$C_{\epsilon mn} := \partial^3_{\epsilon mn}\theta(s+u) = (\bar{T}_\epsilon.T_m.T_n).$$

Denote by $\bar{T}^\epsilon \in H^4(X)$ the dual basis of $\bar{T}_\epsilon$'s, and write

$$T^l, \qquad 1 \le l \le \rho,$$

the dual basis of T_l's. Also $\bar{T}_0 = T_0 = \mathbf{1}$ with dual $\bar{T}^0 = T^0$ the point class.

Remark 3.10. The more canonical choice

$$T^{(l)} := \sum_{i=1}^{k} b_{il}[C_i]$$

is not the dual basis since

$$(T^{(l)}.T_m) = \sum_{i=1}^{k} b_{il}(C_i.T_m) = \sum_{i=1}^{k} b_{il}b_{im} = (B^tB)_{lm}.$$

This implies that

$$T^{(l)} = \sum_{m=1}^{\rho} (B^tB)_{lm} T^m.$$

This canonical basis will be useful later when we discuss the monodromy.

Since

$$H^{ev}(Y) = \iota(H^{ev}(X)) \oplus \Big(\bigoplus_{l=1}^{\rho} \mathbb{Q}T_l \oplus \bigoplus_{l=1}^{\rho} \mathbb{Q}T^l \Big)$$

is an orthogonal decomposition with respect to the Poincaré pairing on $H(Y)$, under our choice of basis we have four types of structural coefficients

$$\begin{aligned} C^{\iota}_{\epsilon\zeta}(s) = C_{\epsilon\zeta\iota}(s), \qquad C^{n}_{lm}(u) = C_{lmn}(u), \\ C^{n}_{\epsilon m} = C_{\epsilon mn}, \qquad C^{\epsilon}_{mn} = C_{\epsilon mn}, \end{aligned} \tag{3.10}$$

where the last two expressions are topological constants.

Now we need to bring back the missing topological terms

$$\frac{1}{2}(s^0)^2 s^{0'} + s^0 \sum_{\epsilon} u^l u^{l'}$$

where we relabel the indices by $u^{l'} = u_l$ and $s^{0'} = s_0$. These give rise to a few more non-trivial constant structural coefficients

$$C_{000'} = 1, \qquad C_{mn'0} = \delta_{mn}.$$

To close this subsection, we notice that in terms of the basis T_l's with coordinates $u^1, \cdots, u^{\rho}$ the degeneration loci D of the GW theory consists of the k hyperplanes defined by

$$D_i := \left\{ v_i := \sum_{p=1}^{\rho} b_{ip} u^p = 0 \right\}.$$

Whenever $\rho > 1$, the divisor $\mathfrak{D} = \bigcup_{i=1}^{k} D_i$ is *not* a normal crossing divisor. Thus in order to study the monodromy effects of the degeneration there are indeed k primitive monodromy transformations $N^{(i)}$'s, which cross D_i's respectively for $1 \le i \le k$ (to be studied in Lemma 3.12). Geometrically D_i is the Kähler degenerating locus at which C_i shrinks to zero volume.

3.3. **The Dubrovin connection.**

3.3.1. The Dubrovin connection on $TH^{ev}(Y)$

$$\nabla^z = d - \frac{1}{z} \sum_{P} dt^P \otimes T_P *$$

with respect to this basis restricts to the Dubrovin connection on $TH^{ev}(X)$. For the other part with basis T_l's and T^l's, we have

$$\begin{aligned} z\nabla^z_{\partial_l} T^m &= -\delta_{lm} T^0, \\ z\nabla^z_{\partial_l} T_m &= -\sum_{n=1}^{\rho} C_{lmn}(u) T^n - \sum_{\epsilon} C_{lm\epsilon} \bar{T}^{\epsilon}, \\ z\nabla^z_{\partial_\epsilon} T_m &= -\sum_{n=1}^{\rho} C_{\epsilon mn} T^n. \end{aligned} \tag{3.11}$$

The second equation shows that the connection does not preserve the subbundle spanned by T_m's and T^n's even along the $u^1, \cdots, u^\rho$ coordinates.

Let

$$\tau = \tau^0 + \tau^1 + \tau^2 \in H^0(Y) \oplus H^2(Y) \oplus H^4(Y) = H^{ev}(Y).$$

The TRR (topological recursive relation) implies that a complete basis of flat sections are given by the derivatives of the big J function

$$J^Y(\tau, z^{-1}) = e^{\frac{\tau^0+\tau^1}{z}} \sum_{\gamma,n,P} \frac{q^\gamma}{n!} e^{(\tau^1.\gamma)} T_P \left\langle \frac{T^P}{z(z-\psi)}, (\tau^2)^n \right\rangle^Y_{0,n+1,\gamma}.$$

That is,

$$z\partial_P\, z\partial_Q J = \sum C^R_{PQ}\, z\partial_R J,$$

known as the quantum differential equation.

Notice that on the H^2 directions we have

$$z\partial_I J = e^{\frac{\tau^0+\tau^1}{z}} \sum_{\gamma,n,P} \Big(T_I + z(\gamma.T_I)\Big) \frac{q^\gamma}{n!} e^{(\tau^1.\gamma)} T_P \left\langle \frac{T^P}{z(z-\psi)}, (\tau^2)^n \right\rangle^Y_{0,n+1,\gamma}.$$

In fact modulo Novikov variables $J(\tau, z^{-1}) \equiv e^{\tau/z}$ and hence $z\partial_I J \equiv T_I\, e^{\tau/z}$ for any $T_I \in H^{ev}(Y)$. Similarly, $z\partial_I\, z\partial_J J \equiv T_I.T_J\, e^{\tau/z}$.

3.3.2. It will be instructive to first study monodromy transformation even though we are seeking for information beyond monodromy. In mirror symmetry, one is interested in "large Kähler structure limit" or "maximal Kähler degeneration" in the sense that $q^\gamma \to 0$ for any $\gamma \neq 0$. From Remark 3.8, we know that when the Novikov variables are omitted, their corresponding analytic properties can be read out from the divisorial parameter τ^1 Let $q_I = \exp t^I$. From the expression of the flat sections $z\partial_P J$'s, the monodromy are all contributed from the exponential factor

$$\exp \tau^1/z = \exp \sum_I t^I T_I/z = \exp \sum_I \frac{\log q_I}{z} T_I.$$

Thus the nilpotent monodromy transformation along the divisor $q_I = 0$ is

$$N_I = \frac{2\pi\sqrt{-1}}{z} T_I \cup.$$

For Calabi–Yau or smooth minimal 3-folds, the big J function is equal to the small J function by virtual dimension count. Equivalently, the corresponding flat sections for the extremal part are easily seen to be given by

$$\begin{aligned}
\sigma^m &= T^m + \frac{1}{z} u^m T^0, \\
\sigma_m &= T_m + \frac{1}{z} \sum C_{mn}(u) T^n + \frac{1}{z} \sum C_{m\epsilon}(u) \bar{T}^\epsilon + \frac{1}{z^2} C_m(u) T^0.
\end{aligned}$$

Here $C_{m\epsilon}(u) = \sum_n C_{\epsilon mn} u^n$ is linear. Other terms are derivatives of $E^Y_0(u)$.

Our case of crepant extremal contraction to ordinary double points corresponds to "small Kähler structure limit" or "minimal Kähler degeneration". The singular part comes from different sources. For the boundary face of the complexified Kähler cone defined by one linear equation $\sum b_I t^I = 0$, we have the corresponding $\prod q_I^{b_I} = 1$. Thus if τ^1 lies in such a boundary face, the summation over γ with $(\tau^1.\gamma) = 0$ may possibly leads to divergent series. Its leading logarithmic term then gives rise to the nilpotent part of the monodromy transformation.

In the original connection form, the nilpotent monodromy $N_l = (N_{l,mn}) \in M_{\rho\times\rho}$ is seen to be $2\pi\sqrt{-1}$ times the residue matrix of the connection. And in our concrete case along $u^l = 0$ it is precisely

$$N_{l,mn} = -\frac{2\pi\sqrt{-1}}{z} \operatorname*{Res}_{q_l=1} C_{lmn}.$$

Example 3.11. In the fundamental special case $\rho = 1$, namely $\psi : Y \to \bar{X}$ is a primitive contraction, we have all $m = n = \rho = 1$ and the nilpotent monodromy is of the form

$$N = N_l = \begin{pmatrix} 0 & N_{l,11} \\ 0 & 0 \end{pmatrix}$$

determined by one entry. There is only one vector $B = B_1 = (b_1, \cdots, b_k)^t$, and for $q := e^t$ we can easily determine the residue as

$$N_{l,11} = -\frac{2\pi\sqrt{-1}}{z} \sum_{i=1}^{k} b_i^3 \operatorname*{Res}_{q=1} \frac{-1}{q^{b_i} - 1} = \frac{2\pi\sqrt{-1}}{z} \sum_{i=1}^{k} b_i^2 = \frac{2\pi\sqrt{-1}}{z} |B_1|^2.$$

Here the Novikov variables are omitted.

Notice that the calculation is still valid even if some $b_i = 0$. However, for Calabi–Yau threefolds this never occurs since the small resolution $Y \to \bar{X}$ exists if and only if there are strictly non-trivial relations among the vanishing spheres in X [34], and for $\rho = 1$ there is precisely only one relation $\sum_{i=1}^k b_i [S_i] = 0$.

In the general case the discriminant loci $\mathfrak{D} = \bigcup_{i=1}^k D_i$ is not a normal crossing divisor. But the monodromy transformation $N^{(i)}$ associated to D_i can be determined in a similar manner. In fact for any $I \subset \{1, \cdots, k\}$ and $D_I := \bigcap_{i\in I} D_i$ we may study the one parameter Kähler degeneration towards D_I and determine its monodromy.

Lemma 3.12. *In terms of $\{T_n\}$ and dual basis $\{T^n\}$, the nilpotent monodromy $N_{(i)}$ along the degenerate divisor D_i defined by $v_i = \sum_{p=1}^{\rho} b_{ip} u^p = 0$ is given by*

$$N_{(i),mn} = \frac{2\pi\sqrt{-1}}{z} b_{im} b_{in}.$$

Proof. Since $\partial_{u^l} = \sum_{i=1}^k (\partial v_i/\partial u^l)\partial_{v_i} = \sum_{i=1}^k b_{il}\partial_{v_i}$, we get

$$N_{(i),mn} = -\frac{2\pi\sqrt{-1}}{z} b_{im} b_{in} \operatorname*{Res}_{v_i=0} \frac{-1}{e^{v_i}-1}$$

which gives the result. □

Corollary 3.13. *In terms of $\{T_n\}$ and dual basis $\{T^n\}$, the nilpotent monodromy N_l at $u=0$ around $q_l := \exp u^l = 1$ is given by*

$$N_l = \frac{2\pi\sqrt{-1}}{z} B_l^t B_l,$$

where B_l be the sub-matrix of B consisting of those i-th rows with $b_{il} \neq 0$.

Proof. This follows from Lemma 3.12. We may also prove it directly. To determine $N_{l,mn}$ along the hyperplane $q_l = 1$ in the Kähler moduli ($u^l = 0$ in the Kähler cone), at the point $q_p = 1$ for all p ($u^p = 0$), we compute

$$\begin{aligned} N_{l,mn} &= -\frac{2\pi\sqrt{-1}}{z} \sum_{i=1}^k b_{il} b_{im} b_{in} \operatorname*{Res}_{q=1} \frac{-1}{q^{b_{il}}-1} \\ &= \frac{2\pi\sqrt{-1}}{z} \sum_{b_{il}\neq 0;\, i=1}^k b_{im} b_{in} = \frac{2\pi\sqrt{-1}}{z} (B_l^t B_l)_{mn}. \end{aligned}$$

□

4. PERIODS AND GAUSS–MANIN CONNECTIONS

From this section and on, unless stated otherwise, we will assume the Calabi–Yau condition:

$$K_X \cong \mathcal{O}_X, \quad H^1(\mathcal{O}_X) = 0.$$

4.1. **Deformation theory.** The main references for this subsection are [14, 30], though we follow the latter more closely.

Let $\Omega_{\bar X}$ be the sheaf of Kähler differential and

$$\Theta_{\bar X} := \mathcal{H}om(\Omega_{\bar X}, \mathcal{O}_{\bar X})$$

be its dual. The deformation of $\bar X$ is governed by $Ext^1(\Omega_{\bar X}, \mathcal{O}_{\bar X})$. By local to global spectral sequence, we have

$$\begin{aligned} 0 \to H^1(\bar X, \Theta_{\bar X}) &\xrightarrow{\lambda} Ext^1(\Omega_{\bar X}, \mathcal{O}_{\bar X}) \\ &\to H^0(\bar X, \mathcal{E}xt^1(\Omega_{\bar X}, \mathcal{O}_{\bar X})) \xrightarrow{\kappa} H^2(\bar X, \Theta_{\bar X}). \end{aligned} \tag{4.1}$$

Since $\mathcal{E}xt^1(\Omega_{\bar X}, \mathcal{O}_{\bar X})$ is supported at the ordinary double points p_i's, we have

$$H^0(\bar X, \mathcal{E}xt^1(\Omega_{\bar X}, \mathcal{O}_{\bar X})) = \bigoplus_{i=1}^k H^0(\mathcal{O}_{p_i})$$

by an easy local computation.

We rephrase the deformation theory on $\bar{X}$ in terms of the log deformation on $\tilde{Y}$. Denote by $E \subset \tilde{Y}$ the union of the exceptional divisors of $\tilde{\psi} : \tilde{Y} \to \bar{X}$.

Lemma 4.1.

$$R\tilde{\psi}_* K_{\tilde{Y}} = \tilde{\psi}_* K_{\tilde{Y}} = K_{\bar{X}}$$

and hence

$$H^0(K_{\tilde{Y}}) \cong H^0(K_{\bar{X}}) \cong \mathbb{C}.$$

Proof. Apply the Serre duality for the projective morphism $\tilde{\psi}$ and we have

$$R\tilde{\psi}_* K_{\tilde{Y}} \cong (\tilde{\psi}_* \mathcal{O}_{\tilde{Y}} \otimes K_{\bar{X}})^\vee.$$

Since $\bar{X}$ is normal rational Gorenstein, $\tilde{\psi}_* \mathcal{O}_{\tilde{Y}} \cong \mathcal{O}_{\bar{X}}$. This proves the first equation from which the first part of the second equation follows immediately. The second part of the second equation follows from the Calabi–Yau condition $K_{\bar{X}} \cong \mathcal{O}_{\bar{X}}$. □

Lemma 4.2.

$$\Omega^2_{\tilde{Y}}(\log E) \cong K_{\tilde{Y}} \otimes \left(\Omega_{\tilde{Y}}(\log E)(-E)\right)^\vee.$$

Proof. On $\tilde{Y}$ there is a perfect pairing

$$\Omega_{\tilde{Y}}(\log E) \otimes \Omega^2_{\tilde{Y}}(\log E) \to K_{\tilde{Y}}(E).$$

Since $\tilde{Y}$ is nonsingular and E is a disjoint union of nonsingular divisors, all sheaves involved are locally free. Hence the lemma follows. □

Lemma 4.3 ([30, Lemma 2.5])**.**

$$L\tilde{\psi}^* \Omega_{\bar{X}} \cong \tilde{\psi}^* \Omega_{\bar{X}} \cong \Omega_{\tilde{Y}}(\log E)(-E),$$

where $L\tilde{\psi}^*$ is the left-derived functor of the pullback map.

Proof. The second isomorphism can be seen by a local calculation of the blowing-up of an ordinary double point. The first isomorphism follows from the facts that $\bar{X}$ is a local complete intersection and an explicit two-term resolution of $\Omega_{\bar{X}}$ exists. We sketch the argument here and refer to [30] for more details. Locally near a node, defined by $x_1^2 + \cdots + x_4^2 = 0$, one has

$$0 \to \mathcal{O} \xrightarrow{2\vec{x}} \mathcal{O}^4 \to \Omega \to 0.$$

Pulling it back to $\tilde{Y}$, we see that

$$\tilde{\psi}^*(2\vec{x}) : \mathcal{O} \to \mathcal{O}^4$$

is injective on Y and therefore higher left-derived functors are zero. □

Lemma 4.4 ([30, Proposition 2.6])**.**

$$R\mathcal{H}om(\Omega_{\bar{X}}, K_{\bar{X}}) \cong R\tilde{\psi}_* \Omega^2_{\tilde{Y}}(\log E).$$

In particular,

$$Ext^1(\Omega_{\bar{X}}, K_{\bar{X}}) \cong H^1(\Omega^2_{\tilde{Y}}(\log E)).$$

Proof. By Lemma 4.2, we have

$$R\tilde{\psi}_*\Omega^2_{\tilde{Y}}(\log E) \cong R\tilde{\psi}_*\mathcal{H}om(\Omega_{\tilde{Y}}(\log E)(-E), K_{\tilde{Y}}).$$

By Lemma 4.3 and the projection formula, the RHS is isomorphic to

$$R\mathcal{H}om(\Omega_{\bar{X}}, R\tilde{\psi}_*K_{\tilde{Y}}) \cong R\mathcal{H}om(\Omega_{\bar{X}}, K_{\bar{X}})$$

with the last isomorphism coming from $R\tilde{\psi}_*K_{\tilde{Y}} \cong K_{\bar{X}}$ in Lemma 4.1. □

From the general deformation theory, the first term $H^1(\bar{X}, \Theta_{\bar{X}})$ in (4.1) parameterizes equisingular deformation of $\bar{X}$. Thanks to the theorem of Kollár and Mori [15] that this extremal contraction deforms in families, this term parameterizes the deformation of Y. Therefore, the cokernel of λ in (4.1), or equivalently the kernel of κ, corresponds to the deformation of singularity. Since the deformation of $\bar{X}$ is unobstructed [14], $\mathrm{Def}(\bar{X})$ has the same dimension as $\mathrm{Def}(X)$, which is $h^{2,1}(X)$. Comparing the Hodge number $h^{2,1}$ of X and $\bar{Y}$ (cf. Section 2) we have the $\dim\ker(\kappa) = \mu$.

Proposition 4.5.

$$0 \to H^1(\bar{X}, \Theta_{\bar{X}}) \xrightarrow{\lambda} Ext^1(\Omega_{\bar{X}}, \mathcal{O}_{\bar{X}}) \to V^* \to 0$$

Proof. The residue exact sequence on $\tilde{Y}$ goes as

$$0 \to \Omega_{\tilde{Y}} \to \Omega_{\tilde{Y}}(\log E) \xrightarrow{\text{res}} \mathcal{O}_E \to 0.$$

Taking wedge product with $\Omega_{\tilde{Y}}$, we have

$$0 \to \Omega^2_{\tilde{Y}} \to \Omega^2_{\tilde{Y}}(\log E) \xrightarrow{\text{res}} \Omega_E \to 0.$$

Part of the cohomological long exact sequence reads

$$H^0(\Omega_E) \to H^1(\Omega^2_{\tilde{Y}}) \to H^1(\Omega^2_{\tilde{Y}}(\log E)) \to H^1(\Omega_E) \xrightarrow{\kappa} H^2(\Omega^2_{\tilde{Y}}).$$

Since $H^1(E) = 0$, the first term vanishes. By Lemma 4.4, the third term is equal to $Ext^1(\Omega_{\bar{X}}, \mathcal{O}_{\bar{X}})$. Indeed, it is not hard to see that this exact sequence is equal to that in (4.1) (cf. [30, (3.2)]).

Using similar arguments in Section 2, especially Section 2.2.2, we have

$$0 \to H^1(\Omega^2_{\tilde{Y}}) \to H^1(\Omega^2_{\tilde{Y}}(\log E)) \to \bigoplus_{i=1}^{k}\langle(\ell_i - \ell'_i)\rangle \xrightarrow{\bar{\kappa}} \frac{H^2(\Omega^2_{\tilde{Y}})}{\bigoplus_{i=1}^{k}\langle(\ell_i + \ell'_i)\rangle}.$$

Recall that, from (2.4) and Lemma 2.6 (ii), we have

$$H^2(\tilde{Y}) \xrightarrow{\bar{\delta}_2} \bigoplus_{i=1}^{k}\langle(\ell_i - \ell'_i)\rangle \to V \to 0.$$

Now compare the dual the map $\bar{\delta}_2$ and $\bar{\kappa}$, we see that

$$\ker(\kappa) = \mathrm{cok}(\bar{\delta}_2)^* = V^*.$$

The proof is complete. □

This proposition shows that the deformation of Y naturally embeds to that of $\bar{X}$, with the transversal direction given by the periods of the vanishing cycles. Moreover, the above discussion also leads to important consequences on the infinitesimal period relations on $\tilde{Y}$ and on $\bar{X}$.

Corollary 4.6. *On $\tilde{Y}$, the natural map*

$$H^1((\Omega_{\tilde{Y}}(\log E)(-E))^\vee) \otimes H^0(K_{\tilde{Y}}) \to H^1(\Omega^2_{\tilde{Y}}(\log E))$$

coming from infinitesimal log deformations of $(\tilde{Y}, E)$ is an isomorphism.

Proof. This follows from Lemma 4.1 and Lemma 4.2. □

Corollary 4.7. *On $\bar{X}$, the natural map*

$$H^1(R\mathcal{H}om(\Omega_{\bar{X}}, \mathcal{O}_{\bar{X}})) \otimes H^0(K_{\bar{X}}) \to Ext^1(\Omega_{\bar{X}}, K_{\bar{X}})$$

coming from infinitesimal deformations of $\bar{X}$ is an isomorphism.

Indeed, both the LHS and RHS are isomorphic to $Ext^1(\Omega_{\bar{X}}, \mathcal{O}_{\bar{X}})$.

Proof. This is a reformulation of Corollary 4.6 via Lemma 4.4. □

Since $\bar{X}$ is rational Gorenstein, $R\mathcal{H}om(\Omega_{\bar{X}}, \mathcal{O}_{\bar{X}})$ has cohomology only in degrees 0 and 1. Indeed, $R^0\mathcal{H}om(\Omega_{\bar{X}}, \mathcal{O}_{\bar{X}}) \cong \Theta_{\bar{X}}$ by definition and

$$R^1\mathcal{H}om(\Omega_{\bar{X}}, \mathcal{O}_{\bar{X}}) \cong \mathcal{E}xt^1(\Omega_{\bar{X}}, \mathcal{O}_{\bar{X}}) \cong \bigoplus_{i=1}^{k} \mathcal{O}_{p_i}.$$

Therefore, by a Leray spectral sequence argument, this gives (4.1) as well and

$$H^1(R\mathcal{H}om(\Omega_{\bar{X}}, \mathcal{O}_{\bar{X}})) \cong Ext^1(\Omega_{\bar{X}}, \mathcal{O}_{\bar{X}}).$$

Interpreting Corollary 4.7 as a local Torelli type theorem, we conclude that the differentiation of the holomorphic 3-forms on any deformation parameter of $\bar{X}$ is non-vanishing.

4.2. **Vanishing cycles and the Bryant–Griffiths/Yukawa cubic form.**

4.2.1. Recall the Gauss–Manin connection ∇^{GM} on the bundle

$$\mathcal{H}^n = R^n f_* \mathbb{C} \otimes \mathcal{O}_S \to S$$

for a smooth family $f : \mathcal{X} \to S$ is a flat connection with its flat sections being identified with the local system $R^n f_* \mathbb{C}$. It contains the integral flat sections $R^n f_* \mathbb{Z}$. Let $\{\delta_p \in H_n(X, \mathbb{Z})/(\text{torsions})\}$ be a homology basis for a fixed reference fiber $X = \mathcal{X}_{s_0}$, with cohomology dual basis δ_p^*'s in $H^n(X, \mathbb{Z})$. Then δ_p^* can be extended to (multi-valued) flat sections in $R^n f_* \mathbb{Z}$. For $\eta \in \Gamma(S, \mathcal{H}^n)$, we may rewrite it in terms of these flat frames with coefficients being the "multi-valued" period integrals "$\int_{\delta_p} \eta$" as

$$\eta = \sum_p \delta_p^* \int_{\delta_p} \eta.$$

Let (x_j) be a local coordinate system in S. Since $\nabla^{GM}\delta_p^* = 0$, we get

$$\nabla^{GM}_{\partial/\partial x_j}\eta = \sum_p \delta_p^* \frac{\partial}{\partial x_j}\int_{\delta_p}\eta.$$

Thus as far as period integrals are concerned, we may simply regard the Gauss–Manin connection as partial derivatives.

When the family contains singular fibers, by embedded resolution of singularities we may assume that the discriminant loci $\mathfrak{D} \subset S$ is a normal crossing divisor. It is well-known that the Gauss–Manin connection has at most regular singularities along $\mathfrak{D}$ by the regularity theorem. Namely it admits an extension to the boundary with at worst logarithmic poles.

4.2.2. We move on to investigate the partial compactification of the complex moduli space $\mathcal{M}_X$ of X towards the conifold degeneration boundary, i.e., in the neighborhood of $[\bar{X}] \in \mathcal{M}_{\bar{X}}$. In particular, we will study the corresponding logarithmic structure of the Gauss–Manin connection near the moduli point $[\bar{X}]$.

Based on Proposition 4.5 and the theory of Bryant and Griffiths, we will show that *periods of vanishing cycles* give rise to a natural coordinate system of the deformations of X in the transversal directions towards the boundary containing the point $[\bar{X}]$ with the same singularity type. On the contrary, the monodromy invariant periods lift to the central fiber by the invariant cycle theorem. The central fiber cohomology is related to $H(Y)$ through the calculation in Section 2. In particular, the invariant Gauss–Manin system gives the Gauss–Manin system on Y. The details of this outline will be spelled out in the rest of the section.

Our setting, as before, is a projective conifold transition $X \nearrow Y$. We have seen that (c.f. Remark 2.10 and the convention following it)

$$H_3(X) \cong H_3(Y) \oplus V \oplus V',$$

where $V \oplus V' \cong H_3(Y)^\perp$, V and V' are isotropic subspaces and are dual to each other under intersection pairing. In particular, $V' \cong V^*$ via Poincaré duality. Recall that $A = (a_{ij}) \in M_{k\times\mu}(\mathbb{Z})$ is the (rank μ) relation matrix of the exceptional curves C_i's. We choose a basis $\{\gamma_j\}_{j=1}^{\mu}$ of V' by requiring that

$$\mathrm{PD}(\gamma_j)([S_i]) \equiv (\gamma_j.S_i) = a_{ij}, \qquad 1 \le j \le \mu,$$

where S_i's are the vanishing 3-spheres. Additionally, let $\{\Gamma_j\}_{j=1}^{\mu}$ be the basis of V dual to $\{\gamma_j\}_{j=1}^{\mu}$ via intersection pairing. Namely $(\Gamma_j.\gamma_l) = \delta_{jl}$.

Remark 4.8. The expression $\Gamma_j = \sum_{i=1}^k c_{ij}[S_i]$ is by no means unique. The more natural choice $\Gamma_{(j)} := \sum_{i=1}^k a_{ij}[S_i]$ does not give the dual. Instead,

$$(\mathrm{PD}(\gamma_l), \Gamma_{(j)}) = \sum_{i=1}^k a_{ij}(\gamma_l.S_i) = \sum_{i=1}^k a_{ij}a_{il} = (A^tA)_{jl}.$$

Lemma 4.9. *We may construct a symplectic basis of $H_3(X)$:*

$$\alpha_0, \alpha_1, \cdots, \alpha_h, \beta_0, \beta_1, \cdots, \beta_h, \qquad (\alpha_j.\beta_p) = \delta_{jp},$$

where $h = h^{2,1}(X)$, with

$$\alpha_j = \Gamma_j, \qquad 1 \le j \le \mu.$$

Proof. Notice that $V \subset H_3(X, \mathbb{Z})$ is generated by $[S_i^3]$'s, and hence is totally isotropic. Let $W \supset V$ be a maximal isotropic subspace (of dimension $h+1$). We first select $\alpha_j = \Gamma_j$ for $1 \le j \le \mu$ to form a basis of V. We then extend it to $\alpha_1, \cdots, \alpha_h$, and set $\alpha_0 \equiv \alpha_{h+1}$, to form a basis of W.

To construct β_l, we start with any δ_l such that $(\alpha_p.\delta_l) = \delta_{pl}$. Such δ_l's exist by the non-degeneracy of the Poincaré pairing. We set $\beta_1 = \delta_1$. By induction on l, suppose that $\beta_1, \cdots, \beta_l$ have been constructed. We define

$$\beta_{l+1} = \delta_{l+1} - \sum_{p=1}^{l} (\delta_{l+1}.\beta_p)\alpha_p.$$

Then it is clear that $(\beta_{l+1}.\beta_p) = 0$ for $p = 1, \cdots, l$. □

With a choice of basis of $H_3(X)$, any element $\eta \in H^3(X, \mathbb{C}) \cong \mathbb{C}^{2(h+1)}$ is identified with its "coordinates" given by the period integrals

$$\vec{\eta} = \Big(\int_{\alpha_p} \eta, \int_{\beta_p} \eta \Big).$$

Alternatively, we denote the cohomology dual basis by α_p^* and β_p^* so that $\alpha_j^*(\alpha_p) = \delta_{jp} = \beta_j^*(\beta_p)$. Then we may write

$$\eta = \sum_{p=0}^{h} \alpha_p^* \int_{\alpha_p} \eta + \beta_p^* \int_{\beta_p} \eta.$$

The symplectic basis property implies that

$$\alpha_p^*(\Gamma) = (\Gamma.\beta_p), \qquad \beta_p^*(\Gamma) = -(\Gamma.\alpha_p) = (\alpha_p.\Gamma).$$

This leads to the following observation.

Lemma 4.10. *For $1 \le j \le \mu$, we may modify γ_j by vanishing cycles to get*

$$\gamma_j = \beta_j.$$

In particular, $(\gamma_j.\gamma_l) = 0$ for $1 \le j, l \le \mu$ and $\alpha_j^*(S_i) = (S_i.\beta_j) = -a_{ij}$.

The following lemma will be useful.

Lemma 4.11. *For all $i = 1, \ldots, k$,*

$$\mathrm{PD}([S_i]) = -\sum_{j=1}^{\mu} a_{ij}\ \mathrm{PD}(\Gamma_j).$$

Proof. Comparing both sides by evaluating at α_l's and β_l's for all l. □

4.2.3. Let Ω be the nonvanishing holomorphic 3-form on the Calabi–Yau threefold. Bryant–Griffiths in [3] showed that the α-periods $x_p = \int_{\alpha_p} \Omega$ form the projective coordinates of the image of the period map inside

$$\mathbb{P}(H^3) \cong \mathbb{P}^{2h+1}$$

as a Legendre sub-manifold of the standard holomorphic contact structure. It follows that under such coordinates there is a holomorphic *pre-potential function* $u(x_0, \cdots, x_h)$, which is homogeneous of weight two, such that

$$u_j \equiv \frac{\partial u}{\partial x_j} = \int_{\beta_j} \Omega. \tag{4.2}$$

In fact,

$$u = \tfrac{1}{2} \sum_{p=0}^{h} x_p u_p = \tfrac{1}{2} \sum_{p=0}^{h} x_p \int_{\beta_p} \Omega. \tag{4.3}$$

Hence

$$\Omega = \sum_{p=0}^{h} (x_p \, \alpha_p^* + u_p \, \beta_p^*).$$

In particular, further differentiations in x_j's lead to

$$\partial_j \Omega = \alpha_j^* + \sum_{p=0}^{h} u_{jp} \, \beta_p^*, \qquad \partial_{jl}^2 \Omega = \sum_{p=0}^{h} u_{jlp} \, \beta_p^*.$$

By the Griffiths transversality, $\partial_j \Omega \in F^2$, $\partial_{jl}^2 \Omega \in F^1$, and all are orthogonal to F^3. Hence we have the *Bryant–Griffiths cubic form*, which is homogeneous of weight -1:

$$u_{jlm} = (\partial_m \Omega . \partial_{jl}^2 \Omega) = \partial_m (\Omega . \partial_{jl}^2 \Omega) - (\Omega . \partial_{jlm}^3 \Omega) = -(\Omega . \partial_{jlm}^3 \Omega).$$

This is also known as *Yukawa coupling* in the physics literature.

We might need to work with moduli parameters which correspond to inhomogeneous coordinates $z_i = x_i / x_0$. The corresponding formulae may be deduced from the homogeneous ones by the following fact:

Lemma 4.12. *For any multi-index I, $\partial^I u$ is homogeneous of weight* $2 - |I|$.

The following proposition shows that, under a suitable choice of the holomorphic frames respecting the Hodge filtration, *the Bryant–Griffiths–Yukawa couplings determine the VHS as the structural coefficients of the Gauss–Manin connection*.

Proposition 4.13. *Consider the successive holomorphic frame $\tau_0 = \Omega \in F^3$, $\tau_j = \partial_j \Omega \in F^2$, $\tau^j = \beta_j^* - (x_j / x_0) \beta_0^* \in F^1$ for $1 \le j \le h$, and $\tau^0 = \beta_0^* \in F^0$.*

Then for $1 \le p, j \le h$,

$$\begin{aligned} \nabla_{\partial_p} \tau_0 &= \tau_p, \\ \nabla_{\partial_p} \tau_j &= \sum_{m=1}^{h} u_{pjm}\, \tau^m, \\ \nabla_{\partial_p} \tau^j &= \delta_{pj}\, \tau^0, \\ \nabla_{\partial_p} \tau^0 &= 0. \end{aligned} \tag{4.4}$$

Proof. We prove the second formula. Since u_{pj} has weight 0, we have the Euler relation $x_0\, u_{pj0} + \sum_{m=1}^{h} x_m\, u_{pjm} = 0$. Hence

$$\begin{aligned} \partial_p \partial_j \Omega &= \sum_{m=1}^{h} u_{pjm}\, \beta_m^* + u_{pj0}\, \beta_0^* \\ &= \sum_{m=1}^{h} u_{pjm} \Big(\beta_m^* - \frac{x_m}{x_0} \beta_0^* \Big) = \sum_{m=1}^{h} u_{pjm}\, \tau^m. \end{aligned}$$

It remains to show that $\tau^j \in F^1$. By the Hodge–Riemann bilinear relations, it is enough to show that $\tau^j \in (F^3)^\perp$. This follows from

$$\begin{aligned} (\tau^j, \Omega) &= \Big(\beta_j^* - \frac{x_j}{x_0} \beta_0^*, \sum_{p=0}^{h} (x_p \alpha_p^* + u_p \beta_p^*) \Big) \\ &= -x_j + \frac{x_j}{x_0} x_0 = 0. \end{aligned}$$

The remaining statements are clear. □

4.3. **Degenerations via Picard–Lefschetz and the nilpotent orbit theorem.** Let $\mathcal{X} \to \Delta$ be a one parameter conifold degeneration of threefolds with nonsingular total space $\mathcal{X}$. Let $S_1, \cdots, S_k$ be the vanishing spheres of the degeneration.. The *Picard–Lefschetz* formula asserts that the monodromy transformation $T : H^3(X) \to H^3(X)$ is given by

$$T\sigma = \sigma + \sum_{i=1}^{k} \sigma([S_i])\, \mathrm{PD}([S_i]), \tag{4.5}$$

where $\sigma \in H^3(X)$. It is *unipotent*, with associated *nilpotent* monodromy

$$N := \log T = \sum_{m=1}^{\infty} (T - I)^m / m.$$

Since S_i has trivial normal bundle in X, we see that $(S_i.S_j) = 0$ for all i, j. In particular $T = I + N$ and $N^2 = 0$. Indeed we have seen these in Section 2 through the Clemens–Schmid exact sequence. The main purpose in this subsection is to extend the discussion to multi-dimensional degenerations, and in particular for the local moduli $\mathcal{M}_{\bar{X}}$ near $[\bar{X}]$.

4.3.1. *VHS with simple normal crossing boundaries.* Even though the discriminant loci for the conifold degenerations under our consideration are in general not SNC divisors, by embedded resolution of singularity they can be modified to become ones. Therefore, we will discuss this case first.

Let

$$\mathcal{X} \to \mathbf{\Delta} := \Delta^{\nu} \times \Delta^{\nu'} \ni \mathbf{t} = (t, s)$$

be a flat family of Calabi–Yau 3-folds such that $X_{\mathbf{t}}$ is smooth for

$$\mathbf{t} = (t, s) \in \mathbf{\Delta}^* := (\Delta^{\times})^{\nu} \times \Delta^{\nu'}.$$

Namely, the discriminant locus is a SNC divisor.

$$\mathfrak{D} := \bigcup_{j=1}^{\nu} Z(t_j) = \mathbf{\Delta} \setminus \mathbf{\Delta}^*.$$

Around each punctured disk $t_j \in \Delta^{\times}$, $1 \leq j \leq \nu$, there is an associated nilpotent monodromy N_j. Let

$$z_j = \log \frac{t_j}{2\pi\sqrt{-1}} \in \mathbb{H}$$

be the coordinates in the upper half plane, and let

$$\mathbf{z}N = \sum_{j=1}^{\nu} z_j N_j.$$

Note that $N_j N_l = N_l N_j$ since $\pi_1(\mathbf{\Delta}^*) \cong \mathbb{Z}^{\nu}$ is abelian.

If for any $\mathbf{t} = (t, s)$ we assume that $X_{\mathbf{t}}$ acquires at most canonical singularities, then $N_j F^3_{\infty}|_{D_j} = 0$ and $N_j^2 = 0$ for each j (c.f. Remark 2.8). Different N_j may define different weight filtration W_j and each boundary divisor $Z(t_j)$ corresponds to different set of vanishing cycles. In our case, the structure turns out to be simple. The degeneration along the curve

$$w \mapsto t(w) = (w^{n_1}, \cdots, w^{n_{\nu}})$$

shows that $(\mathbf{z}N)^2 = 0$ for any $\mathbf{z} \neq 0$. This, together with the *commutativity* of N_j, then implies that $N_j N_l = 0$ for all j, l. For ODP (conifold) degenerations, this is also clear from the Picard–Lefschetz formula (4.5). Indeed $(S_{i_1}.S_{i_2}) = 0$ for all i_1, i_2 implies that $N_j N_l = 0$ for all j, l.

Let Ω denote the relative Calabi–Yau 3-form over $\mathbf{\Delta}$. By Schmid's nilpotent orbit theorem [32] (c.f. [37, 38]), a natural choice of Ω takes the form

$$\begin{aligned} \Omega(\mathbf{t}) = e^{\mathbf{z}N}\mathbf{a}(\mathbf{t}) &= e^{\mathbf{z}N}\Big(a_0(s) + \sum_{j=1}^{\nu} a_j(s) t_j + \cdots\Big) \\ &= \mathbf{a}(\mathbf{t}) + \mathbf{z}N\mathbf{a}(\mathbf{t}) \in F^3_{\mathbf{t}}, \end{aligned} \tag{4.6}$$

where $\mathbf{a}(\mathbf{t})$ is holomorphic, $N_j a_0(s) = 0$ for all j.

In order to extend the theory of Bryant–Griffiths to include the boundary points of the period map, namely to include ODP degenerations in the current case, we need to answer the question if the α-periods

$$\theta_j(\mathbf{t}) := \int_{\Gamma_j} \Omega(\mathbf{t})$$

may be used to replace the degeneration parameters t_j for $1 \le j \le \nu$. For this purpose we need to work on the actual local moduli space $\mathcal{M}_{\bar{X}}$.

4.3.2. *The case of one dimensional degeneration.* It is helpful to first consider the simplest situations that $\nu = \mu = 1$, $\rho = k - 1$. In this case, there is only one (independent) vanishing cycle $\alpha_1 = \Gamma_1 = \Gamma$, $A = (a_{i1}) \in M_{k\times 1}(\mathbb{Z})$ is a column vector, and the degeneration direction is only one dimensional t_1. Then $\theta(t_1, s) = \int_\Gamma \Omega(t_1, s)$ is a continuous, hence holomorphic, parameter.

By Corollary 4.7, $\partial\theta/\partial t_1 = \int_\Gamma \partial_{t_1}\Omega$ is non-zero at $t_1 = 0$. Hence we may use $\mathbf{t} = (\theta, s)$ as the new coordinate system by the implicit function theorem. In terms of this new parameter $t = \theta$, (4.6) leads to

$$\Omega(\mathbf{t}) = a_0(s) + \Gamma^* t + \text{h.o.t.} + \frac{t\log t}{2\pi\sqrt{-1}} N\Gamma^*$$

Here h.o.t. denotes terms in $\Gamma^\perp$ which are at least quadratic in t. Then the first derivatives of the pre-potential function $u_p(\mathbf{t})$ are of the form

$$u_p(\mathbf{t}) = u_p(s) + \text{h.o.t.} + \frac{t\log t}{2\pi\sqrt{-1}} \int_{\beta_p} N\Gamma^*.$$

By the Picard–Lefschetz formula (4.5),

$$N\Gamma^* = \sum_{i=1}^k (\Gamma^*.\,\mathrm{PD}([S_i]))\,\mathrm{PD}([S_i]) = -\sum_{i=1}^k a_{i1}\,\mathrm{PD}([S_i]),$$

where Lemma 4.10 is used in the last equality. Hence

$$\int_{\beta_p} N\Gamma^* = -\sum_{i=1}^k a_{i1}(S_i.\beta_p) = \delta_{1p}\sum_{i=1}^k a_{i1}a_{i1} = \delta_{1p}A^tA = \delta_{1p}|A|^2.$$

In particular, for $p \ne 1$, $u_p(\mathbf{t}) = u_p(s) + \text{h.o.t.}$, and

$$u_t(\mathbf{t}) = u_1(\mathbf{t}) = u_1(s) + \frac{|A|^2}{2\pi\sqrt{-1}} t\log t + \text{h.o.t.}.$$

Therefore the Yukawa coupling is of the expected form

$$u_{ttt}(\mathbf{t}) = \frac{|A|^2}{2\pi\sqrt{-1}}\frac{1}{t} + \text{regular terms}.$$

4.3.3. *Extending the Yukawa coupling towards boundary.* Now we consider the general case. As discussed in Section 4.1 and in the Introduction, $\bar{X}$ has unobstructed deformations and the Kuranishi space $\mathcal{M}_{\bar{X}} = \mathrm{Def}(\bar{X})$ is smooth. Since $\bar{X}$ admits smoothing to X, the dimension of $\mathcal{M}_{\bar{X}}$ is exactly $h = h^{2,1}(X)$. The discriminant loci $\mathfrak{D} \subset \mathcal{M}_{\bar{X}}$ is a divisor, which in general may not be a normal crossing divisor. If we compare with the A model picture on Y, which is discussed in the previous section, the discriminant loci $\mathfrak{D}$ is expected to the union of k hyperplanes.

Recall Friedman's result [7] on partial smoothing of ODP's in the following form. Let $A = [A^1, \cdots, A^\mu]$ be the relation matrix. For any $r \in \mathbb{C}^\mu$, the relation vector

$$A(r) := \sum_{l=1}^{\mu} r_l A^l$$

gives rise to a (germ of) partial smoothing of those ODP's $p_i \in \bar{X}$ with $A(r)_i \neq 0$. Thus for $1 \leq i \leq k$, the linear equation

$$w_i := a_{i1} r_1 + \cdots + a_{i\mu} r_\mu = 0 \tag{4.7}$$

defines a codimension one hyperplane in $\mathbb{C}^\mu$:

$$D^i := Z(w_i)$$

Now the small resolution $\psi : Y \to \bar{X}$ leads to an embedding $\mathcal{M}_Y \subset \mathcal{M}_{\bar{X}}$ of codimension μ. As germs of analytic spaces we thus have

$$\mathcal{M}_{\bar{X}} \cong \Delta^\mu \times \mathcal{M}_Y \ni (r, s).$$

Along each hyperplane $D^i \times \mathcal{M}_Y$, which will still be denoted by D^i, there is a monodromy operator $T^{(i)}$ with associated nilpotent monodromy $N^{(i)} = \log T^{(i)}$. A degeneration from X to X_i with $[X_i] \in D^i$ a general point (not in any other $D^{i'}$ with $i' \neq i$) contains only one vanishing cycle $[S_i^3] \mapsto p_i$. We summarize the above discussion in the following lemma.

Lemma 4.14. *Geometrically a point $(r, s) \in D^i$ corresponds to a partial smoothing X_r of $\bar{X}$ for which the i-th ordinary double point p_i remains singular. Hence, for r generic, the degeneration from X to X_r has only one vanishing sphere S_i^3. Moreover, the Picard–Lefschetz formula says that for any $\sigma \in H^3(X)$,*

$$N^{(i)}\sigma = (\sigma([S_i^3]))\, \mathrm{PD}([S_i^3]).$$

The discriminant locus $\mathfrak{D} = \bigcup_{i=1}^k D^i \subset \mathcal{M}_{\bar{X}}$ is not an SNC divisor, though we can sometimes reduce a problem to the SNC case discussed in Section 4.3.1 by embedded resolution of $(\mathcal{M}_{\bar{X}}, \mathfrak{D})$. For example, let $\mathfrak{D} = \bigcup_{j=1}^{\nu} Z(t_j)$ be an SNC divisor (locally) associated to the embedded resolution such that $\mathbf{t} = (t, s)$ are the coordinates and $Z(t_j)$ the coordinate hyperplanes. If the period integral $\int_\Gamma \Omega$ over a vanishing cycle Γ is a single valued analytic function on $\boldsymbol{\Delta}^*$ then, from (4.6), it admits continuous extensions to $\boldsymbol{\Delta}$ and hence is holomorphic on $\boldsymbol{\Delta}$. This is equivalent to that

$\int_\Gamma N_l \mathbf{a}(\mathbf{t}) = 0$ for all l, which follows easily from the mixed Hodge diamond for N_l:

$$H^3(X) = V_l^* \oplus V_l^\perp, \quad N_l V_l^\perp = 0, \quad N_l V_l^* \subset V_l^\perp, \tag{4.8}$$

where $V_l = \mathbb{C}\Gamma_l$. Since this works for any local chart in the embedded resolution, the conclusion also hold for a relative Calabi–Yau 3-form over $\mathcal{M}_{\bar{X}}$ in the original coordinate system (r, s). Namely, for any vanishing cycle Γ, we get the holomorphicity of the α-period $\int_\Gamma \Omega(r, s)$.

However, we will need more precise information. As in the case of one dimensional degenerations, we look for a formula of Ω in terms of holomorphic $\mathbf{a}(r, s)$, c.f. (4.6) in the case of one-dimensional degeneration. Instead of trying to reduce the problem to the SNC case, we will analyze it directly by extending the nilpotent orbit theorem to the current non-SNC case.

Following the general convention we call the configuration $\mathfrak{D} = \bigcup_{i=1}^k D^i \subset \mathcal{M}_{\bar{X}}$ a *central hyperplane arrangement* with axis $\mathcal{M}_Y$.

Theorem 4.15. *Consider a degeneration of Hodge structures over $\Delta^\mu \times M$ with discriminant locus $\mathfrak{D} = \bigcup_{i=1}^k D^i$ being a central hyperplane arrangement with axis M. Let $N^{(i)}$ be the nilpotent monodromy around the hyperplane $D^i = Z(w_i)$ and suppose that the monodromy group Γ generated by $N^{(i)}$'s is* abelian. *Let D denote the period domain and $\check{D}$ its compact dual.*

Then the period map

$$\phi : \Delta^\mu \times M \setminus \mathfrak{D} \to D/\Gamma$$

takes the following form

$$\phi(r, s) = \exp\left(\sum_{i=1}^k \frac{\log w_i}{2\pi\sqrt{-1}} N^{(i)}\right) \psi(r, s),$$

where $\psi : \Delta^\mu \times M \to \check{D}$ is holomorphic and horizontal.

In particular this applies to degenerations over $\mathcal{M}_{\bar{X}}$ associated to conifold transitions $X \nearrow Y$ of Calabi–Yau 3-folds through the Calabi–Yau conifold $\bar{X}$.

Proof. We prove the theorem by induction on $\mu \in \mathbb{N}$. The case $\mu = 1$ is essentially the one variable case (or SNC case) of the nilpotent orbit theorem. The remaining proof consists of a careful bookkeeping on Schmid's derivation of the multi-variable nilpotent orbit theorem from the one variable case (c.f. [32, § 8], especially Lemma (8.34) and Corollary (8.35)).

The essential statement is the holomorphic extension of

$$\psi(r, s) := \exp\left(-\sum_{i=1}^k \frac{\log w_i}{2\pi\sqrt{-1}} N^{(i)}\right) \phi(r, s) \in \check{D} \tag{4.9}$$

over the locus $\mathfrak{D}$. For $p \notin \{0\} \times M$, we can find a neighborhood U_p of p so that the holomorphic extension to U_p is achieved by induction. Notice that the commutativity of $N^{(i)}$'s is needed in order to arrange $\psi(r,s)$ into the form (4.9) with smaller μ. Namely,

$$\psi = \exp\left(-\sum_{w_i(p)=0} \frac{\log w_i}{2\pi\sqrt{-1}} N^{(i)}\right)\left[\exp\left(-\sum_{w_i(p)\neq 0} \frac{\log w_i}{2\pi\sqrt{-1}} N^{(i)}\right)\phi\right].$$

Let $R_{\geq 1/2} := \{\, (r,s) \mid |r| \geq \frac{1}{2}\,\}$. Then we have a unique holomorphic extension of ψ over $R_{\geq 1/2}$. Now by the Hartog's extension theorem we then get the holomorphic extension to the whole space $\Delta^\mu \times M$. The statement on horizontality follows from the same argument in [32, § 8]. □

In the case of conifold transition, the monodromy group is abelian. This is can be easily seen by the Picard–Lefschetz formula (4.5) and the fact $[S_i].[S_{i'}] = 0$ for all i and i' for the vanishing spheres. Thus Theorem 4.15 is applicable.

Remark 4.16. Let $\mathfrak{D} = \bigcup_{i=1}^k D^i \subset \mathbb{C}^\mu$ be a central hyperplane arrangement with axis 0. Then $\mathbb{C}^\mu \setminus \mathfrak{D}$ can be realized as $(\mathbb{C}^\times)^k \cap L$ for $L \subset \mathbb{C}^k$ being a μ dimensional subspace. Since $\pi_1((\mathbb{C}^\times)^k) \cong \mathbb{Z}^k$, a hyperplane theorem argument shows that $\pi_1(\mathbb{C}^\mu \setminus \mathfrak{D}) \cong \mathbb{Z}^k$, hence abelian, if $\mu \geq 3$.

For $\mu = 2$, $\pi_1(\mathbb{C}^2 \setminus \mathfrak{D})$ is not abelian if $k \geq 3$. Indeed, the natural $\mathbb{C}^\times$ fibration $\mathbb{C}^2 \setminus \bigcup_{i=1}^k D^i \to \mathbb{P}^1 \setminus \{p_1, \cdots, p_k\}$ leads to

$$0 \to \pi_1(\mathbb{C}^\times) \cong \mathbb{Z} \to \pi_1(\mathbb{C}^2 \setminus \bigcup D^i) \to \mathbb{Z}^{*(k-1)} \to 0,$$

where the RHS is a $k-1$ free product of $\mathbb{Z}$. Thus the commutativity assumption on Γ in Theorem 4.15 for $\mu = 2$ might not be superfluous.

Remark 4.17. By studying the transformation of monodromy in the exponential factor under a blowing-up, Theorem 4.15 can also be proved through embedded resolutions to reduce the problem to the SNC case. We leave the details to the interested readers.

Proposition 4.18. *In a neighborhood of $[\bar{X}] \in \mathcal{M}_{\bar{X}}$, we may choose the coordinate system (r,s) so that s is a coordinate system of $\mathcal{M}_Y$ near $[\bar{X}]$ and $r_j = \int_{\Gamma_j} \Omega$, $1 \leq j \leq \mu$, are the α-periods of the vanishing cycles.*

Moreover, the section $\Omega(r,s)$ takes the form

$$\Omega = a_0(s) + \sum_{j=1}^{\mu} \Gamma_j^* r_j + \text{h.o.t.} - \sum_{i=1}^{k} \frac{w_i \log w_i}{2\pi\sqrt{-1}} \,\mathrm{PD}([S_i]).$$

Here h.o.t. denotes terms in $V^\perp$ which are at least quadratic in $r_1, \cdots, r_\mu$, and $w_i = a_{i1} r_1 + \cdots + a_{r\mu} r_\mu = \int_{S_i} \Omega$ defines the discriminant locus D^i for $1 \leq i \leq k$.

Proof. By Theorem 4.15 and the fact $N^{(i_1)}N^{(i_2)} = 0$, we may write

$$\begin{aligned} \Omega(r,s) &= \exp\left(\sum_{i=1}^{k} \frac{\log w_i}{2\pi\sqrt{-1}} N^{(i)}\right) \mathbf{a}(r,s) \\ &= \mathbf{a}(r,s) + \sum_{i=1}^{k} \frac{\log w_i}{2\pi\sqrt{-1}} N^{(i)} \mathbf{a}(r,s) \in F^3_{(r,s)}, \end{aligned} \tag{4.10}$$

where $\mathbf{a}(r,s) = a_0(s) + \sum_{j=1}^{\mu} a_j(s)\, r_j + O(r^2)$ is holomorphic in r, s. Since the integral $\int_{\Gamma_l}$ vanishes on the sum (reasoning as in (4.8)), we have

$$\theta_j := \int_{\Gamma_l} \Omega = \int_{\Gamma_l} \mathbf{a} = \sum_{j=1}^{\mu} \Big(\int_{\Gamma_l} a_j(s) \Big)\, r_j + O(r^2).$$

By Corollary 4.7, the $\mu \times \mu$ matrix

$$(\tau_{lj}(s)) := \Big(\int_{\Gamma_l} a_j(s) \Big)$$

is invertible for all s. Thus, $\theta_1, \cdots, \theta_\mu$ and s form a coordinate system.

Now we replace r_j by the α-period θ_j for $j = 1, \ldots, \mu$. In order for Theorem 4.15 being applicable, we need to justify that the discriminant locus D^i is still defined by linear equations in r_j's. This follows form Lemma 4.11

$$\int_{S_i} \Omega = (\Omega, \mathrm{PD}([S_i])) = -\sum_{j=1}^{\mu} a_{ij} (\Omega, \mathrm{PD}(\Gamma_j)) = -\sum_{j=1}^{\mu} a_{ij} r_j =: -w_j.$$

Denote by h.o.t be terms in $V^\perp$ which are at least quadratic in r_j's. The above choice of coordinates implies that

$$\Omega = a_0(s) + \sum_{j=1}^{\mu} \Gamma_j^* r_j + \text{h.o.t.} + \sum_{i=1}^{k} \sum_{j=1}^{\mu} \frac{\log w_i}{2\pi\sqrt{-1}} N^{(i)} \Gamma_j^* r_j.$$

Then by Lemma 4.14 and 4.10,

$$\sum_{j=1}^{\mu} N^{(i)} \Gamma_j^* r_j = -\sum_{j=1}^{\mu} a_{ij} r_j \, \mathrm{PD}([S_i]) = -w_i \, \mathrm{PD}([S_i]).$$

The proof is complete. □

In terms of the above coordinate system (r,s), we have β-periods

$$u_p(r,s) = \int_{\beta_p} \Omega = u_p(s) + \text{h.o.t.} - \sum_{i=1}^{k} \frac{w_i \log w_i}{2\pi\sqrt{-1}} \int_{\beta_p} \mathrm{PD}([S_i])$$

(since $\Omega(s) = a_0(s)$). For $1 \le p \le \mu$ we get

$$u_p(r,s) = u_p(s) + \text{h.o.t.} + \sum_{i=1}^{k} \frac{w_i \log w_i}{2\pi\sqrt{-1}} a_{ip}.$$

Otherwise we get simply

$$u_p(r, s) = u_p(s) + \text{h.o.t.}$$

The asymptotic of the Yukawa coupling is then completely determined by taking two more partial derivatives.

For example, for $1 \le p, m, n \le \mu$,

$$\begin{aligned} u_{pm} &= O(r) + \sum_{i=1}^{k} \frac{\log w_i + 1}{2\pi\sqrt{-1}} a_{ip} a_{im}, \\ u_{pmn} &= O(1) + \sum_{i=1}^{k} \frac{1}{2\pi\sqrt{-1}} \frac{1}{w_i} a_{ip} a_{im} a_{in}. \end{aligned} \tag{4.11}$$

4.3.4. *Monodromy calculations.* As a simple consequence, we determine the monodromy $N(l)$ towards the coordinate hyperplane $Z(r_l)$ at $r = 0$. That is the monodromy associated to the one parameter degeneration $\gamma(r)$ along the r_l-coordinate axis ($r_l \in \Delta$ and $r_j = 0$ if $j \ne l$). Let $I_l = \{i \mid a_{il} \ne 0\}$ and let A_l be the sub matrix of A consisting of those i-th rows with $i \in I_l$.

Lemma 4.19. *The sphere S_i^3 vanishes in $Z(r_l)$ along transversal one parameter degenerations γ if and only if $i \in I_l$, i.e., $a_{il} \ne 0$.*

Proof. The curve γ lies in $D^i = Z(w_i)$ if and only if $a_{il} = 0$. Thus for those $i \notin I_l$, the ODP p_i is always present on $X_{\gamma(r)}$ along the curve γ. In particular the vanishing spheres along γ are precisely those S_i with $i \in I_l$. □

To calculate the monodromy $N(l)$, recall that (c.f. Lemma 4.10)

$$\Gamma_j^* \equiv \alpha_j^* = -\,\mathrm{PD}(\beta_j), \qquad \Gamma_{(j)} = \sum_{i=1}^{k} a_{ij} [S_i].$$

The Picard–Lefschetz formula (Lemma 4.14) then says that

$$N(l)\Gamma_j^* = \sum_{i \in I_l} (\Gamma_j^*.\,\mathrm{PD}([S_i]))\,\mathrm{PD}([S_i]) = -\sum_{i \in I_l} a_{ij}\,\mathrm{PD}([S_i]).$$

Remark 4.20. If $I_l = \{1, \cdots, k\}$ consists of all the indices, then in terms of the "more canonical choice" of basis $\Gamma_{(j)}$'s of V we get

$$N(l)\Gamma_j^* = -\,\mathrm{PD}(\Gamma_{(j)}).$$

By column and row operations, it is always possible to arrange the relation matrix A so that this holds for all l; namely $A_l = A$ for all l.

For $1 \le p \le \mu$,

$$\int_{\beta_p} N(l)\Gamma_j^* = -\sum_{i \in I_l} a_{ij} (S_i.\beta_p) = \sum_{i \in I_l} a_{ij} a_{ip} = (A_l^t A_l)_{jp},$$

while for $p = 0$ or $\mu + 1 \le p \le h$ we have

$$\int_{\beta_p} N(l)\Gamma_j^* = 0.$$

4.3.5. *On framed logarithmic extensions.* Under suitable choice of frame the Yukawa couplings are the structural coefficients of the Gauss–Manin connection. We will extend this result to the logarithmic Gauss–Manin connection associated to our conifold degenerations.

We seek for a frame of the bundle $R^3\pi_*\mathbb{C}$ of a local family $\pi : \mathcal{X} \to \mathcal{M}_{\bar{X}}$ near the Calabi–Yau conifold $[\bar{X}]$. By Lemma 2.6 and the Hodge diamond (2.9), part of the frame comes naturally from $H^3(Y)$, while the remaining part is modeled on V^* and V. To determine ∇^{GM} (instead of the finer VHS structure), we only need a topological (non-holomorphic) frame.

By the same procedure as in the proof of Proposition 4.18, the topological frame modeled on $V^* \cong H^{2,2}_\infty H^3$ can be chosen to be

$$\begin{aligned} v_j &:= \exp\left(\sum_{i=1}^k \frac{\log w_i}{2\pi\sqrt{-1}} N^{(i)}\right)\Gamma_j^* \\ &= \Gamma_j^* + \sum_{i=1}^k \frac{\log w_i}{2\pi\sqrt{-1}} N^{(i)}\Gamma_j^* = \Gamma_j^* - \sum_{i=1}^k \frac{\log w_i}{2\pi\sqrt{-1}} a_{ij}\,\mathrm{PD}([S_i])\end{aligned} \tag{4.12}$$

for $1 \le j \le \mu$. Notice that the correction terms lie in the lower weight piece $H^{1,1}_\infty H^3$ and v_j is independent of s. Moreover, v_j is singular along D^i if and only if $a_{ij} \ne 0$, i.e., S_i vanishes in $Z(r^j)$ by Lemma 4.19.

On $V \cong H^{1,1}_\infty H^3$, we choose the (constant) frame by

$$v^j := \exp\left(\sum_{i=1}^k \frac{\log w_i}{2\pi\sqrt{-1}} N^{(i)}\right)\mathrm{PD}(\Gamma_j) = \mathrm{PD}(\Gamma_j), \qquad 1 \le j \le \mu. \tag{4.13}$$

From (4.7), (4.12) and Lemma 4.11, it is easy determine the Gauss–Manin connection on this partial frame over the special directions $\partial/\partial r_p$'s:

$$\begin{aligned} \nabla^{GM}_{\partial/\partial r_p} v_m &= \frac{1}{2\pi\sqrt{-1}} \sum_{i=1}^k \frac{a_{ip}}{w_i}\Big(-a_{im}\,\mathrm{PD}([S_i])\Big) \\ &= \frac{1}{2\pi\sqrt{-1}} \sum_{i=1}^k \sum_{n=1}^\mu \frac{a_{ip}a_{im}a_{in}}{w_i}\, v^n.\end{aligned} \tag{4.14}$$

Proposition 4.21. *Near $[\bar{X}] \in \mathcal{M}_{\bar{X}}$, ∇^{GM} is regular singular along D^i's and smooth elsewhere. The connection matrix P on the block $V^* \oplus V$ takes the form*

$$P = \sum_{i=1}^k \frac{dw_i}{w_i} \otimes P^i$$

where

$$P_i = \sum_{m,n=1}^\mu a_{im}a_{in}\, v^n \otimes (v_m)^*$$

is a constant matrix in the topological frame v_m's and v^n's.

In contrast to (4.11), there are no higher order terms in r_j's. Moreover, the off diagonal component of ∇^{GM} from $V^* \oplus V$ to $H^3(Y)$ vanishes. These will be restored in Section 6 once *holomorphic frames* are considered.

5. FROM $A(X) + B(X)$ TO $A(Y) + B(Y)$

We explained in the last section that the Kuranishi space of Y can be identified with the the equisingular part of the Kuranishi space of $\bar{X}$. *We assume that the moduli point $[X]$ is in the neighborhood of $\mathcal{M}_Y$.* The main task of this section is to show that the A theory and B theory on X will determine the A theory and B theory on Y.

For this paper, $A(X)$ means the (all genus) Gromov–Witten theory on X; $B(X)$ means the variations of Hodge structure, or the Gauss–Manin connection, on X (which is in a sense only genus zero part of the B theory).

5.1. **Overview.**

5.1.1. $B(X) \Rightarrow B(Y)$. This was already explained: The VHS on Y is contained in the VHS of X as part of the monodromy invariant theory under the complex degeneration. This is the easy part of the implication.

5.1.2. $A(X) + B(X) \Rightarrow A(Y)$. The rest of the section will be devoted to this implication.

What we already know about $A(Y)$ consists of the following three pieces of data:

(1) $A(X)$, which is given,
(2) the extremal ray invariants on divisors $\{T_l\}_{l=1}^{\rho}$ determined by the relation matrix B of the vanishing 3-spheres, and
(3) the topological cup product on $H^2(Y)$. Since Y comes from surgeries on X along the vanishing spheres, this is determined classically.

We want to see if these are enough to determine $A(Y)$. Indeed, from the degeneration formula derived in Proposition 3.1, the major problem is that the Gromov–Witten invariants on X can only determine a sum of Gromov–Witten invariants on Y, but not the individual terms.

The ingredient (2) is, understandably, the extra information one has to compute separately. As discussed in Section 3.2.2 for $g = 0$ case, the extremal ray invariants of all genera can be obtained from invariants of the a single $(-1,-1)$ curves by the relation matrix A. Therefore, the ingredients needed for (2) is local and independent of the transition. The genus zero case was already discussed. The corresponding $g = 1$ invariants for $(-1,-1)$ curves was computed in [2] (which was justified in [10]) and $g \geq 2$ invariants in [6].

Before proceeding to details, let us look into the genus zero case more closely. Recall that

$$H^{ev}(Y) = \phi^* H^{ev}(X) \oplus \langle T_1, \cdots T_\rho, T^1, \cdots, T^\rho \rangle.$$

Let $\tilde{s} + \tilde{u} \in H(Y)$ be a class in Y according to the above splitting, and Let s (resp. u) be the H^2 component of $\tilde{s}$ (resp. $\tilde{u}$). The GW potential for the Calabi–Yau 3-fold Y takes the form:

$$F_0^Y(\tilde{s} + \tilde{u}) = \frac{(\tilde{s} + \tilde{u})^3}{3!} + \sum_{\gamma \in NE(Y) \setminus \{0\}} n^Y_{\beta + d\ell} \, e^{(\beta.s)} e^{(d\ell.u)}$$

where $\gamma = \beta + d\ell \in NE(Y)$ with $\beta := \psi_*(\gamma) \in NE(X)$ being identified with its canonical lift to Y so that $(\beta.u) = 0$ for all $u = \sum u^l T_l$, and $d\ell := \sum_i d_i [C_i]$. (We omit the Novikov variables, cf. Remark 3.8.)

Notice that while d_i is not uniquely determined by γ, the sum $\sum d_i[C_i]$ is unique as a class. Also

$$(d\ell.u) = \sum_{i,l} (d_i [C_i].u^l T_l) = \sum_{i,l} d_i b_{il} u^l = \vec{d}^t B \vec{u}.$$

Now, the above (1)–(3) gives the initial conditions on the two set of coordinates slices $u = 0$ and "$s = \infty$" (i.e., $\beta = 0$) respectively.

Naively one may wish to reconstruct the genus zero GW theory on the entire cohomology from these two slices. When Y is Fano, this is sometimes possible by WDVV. However, WDVV gives no information for Calabi-Yau 3-folds. Thus, we have to find another way to solve the problem. This issue will be resolved by studying the notion of linking data below.

5.2. **Linking data.** The homology and cohomology discussed in this subsection are over $\mathbb{Z}$.

As a first step, we study the topological information about the holomorphic curves in $X \setminus \bigcup_{i=1}^k S_i$ instead of in X. This can be interpreted as the linking data between the curve C and the set of vanishing spheres $\bigcup_{i=1}^k S_i$. We will see that the linking data add extra information to $\beta = [C]$, and enable us to recover the missing topological information in the process of transition.

The reasons justifying this study are the following heuristics. It is a well known result due to Seidel and Donaldson [33] that the vanishing sphere S_i can be chosen to be Lagrangian with respect to the prescribed Kähler form ω on X. When ω is Ricci flat, it is expected to be able to find special Lagrangian (SL) representatives for S_i. Assuming that, we have

$$T_{[S_i]} \mathrm{Def}(S_i/X) \cong H^1(S_i, \mathbb{R}) = 0$$

by McLean's theorem [25]. That is, S_i is rigid in the SL category. Thus, given a holomorphic curve $C \subset X$ with $[C] = \beta$, we expect that

$$C \cap S_i = \emptyset \quad \forall i.$$

Furthermore, by a simple virtual dimensional count, this is known to hold for a generic almost complex structure J on TX (cf. [8]). But we shall proceed by ignoring this technical issue for a moment.

The plan is to assign a *linking data* L between C and S_i's so that L represents a refinement of $\beta = [C]$ in X and that L uniquely determines a curve class γ in Y, such that

$$n^X_{\beta,L} = n^Y_\gamma. \tag{5.1}$$

With the choices of lifting β in Y being fixed (as above), this is equivalent to saying that L will uniquely determine a curve class $d\ell \in N_1(Y/\bar{X})$.

Remark 5.1. One possible way of defining the linking number is as follows. Let

$$R = \ker(\bigoplus_{i\in I} \mathbb{Z}S_i \to H_3(X,\mathbb{Z})) \subset \mathbb{Z}^I$$

be the relation subgroup of the given spheres. For each $r \in R$ we have

$$r = \sum_{i\in I} r_i S_i = \partial W$$

for some (non-unique) four dimensional chain W. The choices of W is only up to elements in $H_4(X,\mathbb{Z})$. Let $(C.H_4(X,\mathbb{Z})) = m_{[C]}\mathbb{Z}$. Then we may hope to define the linking number through the residue intersection pairing

$$L_C(r) \equiv L(C,r) := \#(C.W) \pmod{m_{[C]}}.$$

Notice that $m_{[C]} = m_\beta$ depends only on the class $\beta = [C]$ but the map $L_C : R \to \mathbb{Z}/m_\beta$ does depend on the specific C, not just on its class, hence it can be used to refine the curve class. However, the mod $m_{[C]}$ is a major drawback as $m_{[C]}$ might be 1. In that case, L_C gives no information. Instead, we proceed by giving the homological description of the above procedure.

Let $D_i = D_\epsilon(N_{S_i/X})$ be the ϵ open tubular neighborhood of S_i in X with ϵ small enough such that $C \cap D_i = \emptyset$ for all i. Then

$$\partial D_i = S_\epsilon(N_{S_i/X}) \cong S_i \times S^2_\epsilon \cong S^3 \times S^2.$$

Let M be the manifold with boundary with $D_I := \bigcup_{i\in I} D_i$ being removed:

$$M = M_I := X \backslash D_I.$$

Remark 5.2. The Lagrangian spheres S_i's are allowed to intersect with each other, though those $S_1, \cdots, S_k$ which vanish in the conifold transition $X \nearrow Y$ are pairwise disjoint (since the nodes are isolated).

We start with the simple case that $I = \{1, \cdots, k\}$. Then the pair $(M, \partial M)$ is the common part for both X and Y. Indeed let $D_i^+ = D_\delta(N_{C_i/Y})$, then

$$\partial D_i^+ = S_\delta(N_{C_i/Y}) \cong S^3_\delta \times C_i \cong S^3 \times S^2.$$

This leads to two deformation retracts

$$(Y, \bigcup C_i) \sim (M, \partial M) \sim (X, \bigcup S_i).$$

Now we consider the sequence induced by the Poincaré–Lefschetz duality and excision theorem for $i : \partial M \hookrightarrow M$ (all with $\mathbb{Z}$ coefficients)

$$
\begin{array}{ccccc}
 & & H_2(M,\partial M) & \xrightarrow{\sim} & H^4(M) \\
 & & \uparrow j_* & & \uparrow j^* \\
H_2(C) & \xrightarrow{f_*} & H_2(M) & \xrightarrow{\sim} & H^4(M,\partial M) \\
 & & \uparrow i_* & & \uparrow \Delta^* \\
 & & \bigoplus_i H_2(S_i^3 \times S_i^2) & \xrightarrow{\sim} & H^3(\partial M) \\
 & & \uparrow \Delta_* & & \uparrow i^* \\
 & & H_3(M,\partial M) & \xrightarrow{\sim} & H^3(M).
\end{array} \tag{5.2}
$$

From the retract $(M, \partial M) \sim (Y, \bigcup C_i)$ and the excision sequence for $(Y, \bigcup C_i)$ we find

$$H_3(M, \partial M) \to \bigoplus H_2(C_i) \to H_2(Y) \to H_2(M, \partial M) \to 0.$$

By comparing this with the LHS vertical sequence we conclude by the five lemma that

$$H_2(M) \cong H_2(Y).$$

In particular, the curve class in Y

$$\gamma := f_*[C] \in H_2(M) \cong H_2(Y)$$

is well defined.

Definition 5.3. The linking data (β, L) is defined to be $f_*([C]) = \gamma$ above.

For any non-canonical splitting of $H_2(Y)$ by a section of j_*, we may represent $\gamma = \beta^Y + d\ell$ where the expression $d\ell = \sum d_i \ell_i$ is only well defined up to curve relations coming from $H_3(Y, \bigcup C_i)$.

From the excision sequence $(X, \bigcup S_i)$, we have

$$0 \to H^3(M, \partial M) \to H^3(X) \to \bigoplus H^3(S_i) \to H^4(M, \partial M) \to H^4(X) \to 0,$$

where the retract $(M, \partial M) \sim (X, \bigcup S_i)$ is used. Comparing with the right vertical sequence in (5.2), we find

$$H^4(M) \cong H^4(X)$$

and $h^3(X) = h^3(M) + k - \rho = h^3(M) + \mu$. Since $h^3(X) = h^3(Y) + 2\mu$, this is equivalent to

$$h^3(M) = h^3(Y) + \mu. \tag{5.3}$$

Remark 5.4. Here we relate γ to the proposal in Remark 5.1. Now each nonzero $r \in R$ is an element in $Z_4(X, \bigcup S_i)$ which is not in $Z_4(X)$. Indeed it corresponds to an equivalence class in

$$\frac{Z_4(X, \bigcup S_i)}{Z_4(X)} \cong \frac{Z_4(X, \bigcup S_i)/B_4(X)}{Z_4(X)/B_4(X)} \cong \frac{H_4(X, \bigcup S_i)}{j_* H_4(X)}.$$

Hence

$$R^* \cong \ker j^* \cong H^3(\bigcup S_i)/i^* H^3(X).$$

Thus the RHS vertical sequence gives a homological interpretation of R.

Since $\gamma = f_*[C] \in H_2(Y) \cong H^4(X, \bigcup S_i)$ and γ is clearly a torsion free class, by the universal coefficient theorem $\gamma \in \mathrm{Hom}(H_4(X, \bigcup S_i), \mathbb{Z})$. From

$$H_4(X, \bigcup S_i) \twoheadrightarrow H_4(X, \bigcup S_i)/j_* H_4(X) \cong R,$$

or equivalently $R^* \cong \ker j^*$ as shown above, we see that γ should give rise to a functional on R only if it restricts to zero values over $H_4(X)$. This is surely not possible by the non-degeneracy of the Poincaré pairing. One way to make this possible is to mod out the values $(C.H_4(X)) = m_\beta \mathbb{Z}$ as we have just done. But in general the above discussion suggest that the curve C in X and its associated class $\gamma = f_*[C]$ in Y already represent the correct notion of "linking data" on both sides respectively.

Remark 5.5. More generally, let

$$\mathbf{S} := \{S_i \mid i \in I\} \supset \{S_1, \cdots, S_k\}$$

be any given finite set of Lagrangian spheres containing the vanishing spheres in X, and let

$$\mathbf{S}' := \{S'_{i'} \mid i' \in I' := I \backslash \{1, \cdots, k\}\}$$

with $S'_{j'}$ being the corresponding Lagrangian sphere in Y. It is clear that the above procedure defines a more general linking data $L_{\mathbf{S}}$ on X and $L'_{\mathbf{S}'}$ on Y. We also expect that

$$n^X_{\beta, L_{\mathbf{S}}} = n^Y_{\gamma, L'_{\mathbf{S}'}}$$

to hold in the more general setting. Indeed, the proof of this is the same as that of (5.1), which will be given later in this section.

5.3. **Linked GW invariants on X = non-extremal GW invariants on Y.**

5.3.1. *Analysis of the moduli of stable maps to the degenerating families.* Here we recall some results in J. Li's study of degeneration formula [20, 21]. Given a projective flat family over a curve

$$\pi : W \to \mathbb{A}^1$$

such that π is smooth away from $0 \in B$ and the central fiber $W_0 = Y_1 \cup Y_2$ has only double point singularity with $D := Y_1 \cap Y_2$ a smooth (but not necessarily connected) divisor, Li constructed a moduli stack in [20]

$$\mathfrak{M}(W, \Gamma) \to \mathbb{A}^1$$

which has a perfect obstruction theory and hence a virtual fundamental class $[\mathfrak{M}(W, \Gamma)]^{\mathrm{virt}}$ in [21]. The following properties will be useful to us. (The notations are slightly changed.)

(1) For every $0 \neq t \in \mathbb{A}^1$, one has

$$\mathfrak{M}(W,\Gamma)_t = \overline{M}(X,\beta), \qquad [\mathfrak{M}(W,\Gamma)]^{\text{virt}}_t = [\overline{M}(X,\beta)]^{\text{virt}}$$

where $\overline{M}(X,\beta)$ is the corresponding moduli of (absolute) stable maps.

(2) For the central fiber, the perfect obstruction theory on $\mathfrak{M}(W,\Gamma)$ induces a perfect obstruction theory on $\mathfrak{M}(W_0,\Gamma)$ and

$$[\mathfrak{M}(W_0,\Gamma)]^{\text{virt}} = [\mathfrak{M}(W,\Gamma)]^{\text{virt}} \cap \pi^{-1}(0)$$

is a virtual divisor of $[\mathfrak{M}(W,\Gamma)]^{\text{virt}}$.

(3) $\mathfrak{M}(W_0,\Gamma)$ and its virtual class are related to the relative moduli and their virtual classes. For each admissible triple (consisting of gluing data) ϵ, there is a "gluing map"

$$\Phi_\epsilon : \mathfrak{M}(Y_1, D; \Gamma_1) \times_{D^\rho} \mathfrak{M}(Y_2, D; \Gamma_2) \to \mathfrak{M}(W_0,\Gamma),$$

inducing the relation between the virtual cycles

$$[\mathfrak{M}(W_0,\Gamma)]^{\text{virt}} = \sum_\epsilon m_\epsilon \Phi_{\epsilon *}\Delta^! \left([\mathfrak{M}(Y_1, D; \Gamma_1)]^{\text{virt}} \times [\mathfrak{M}(Y_2, D; \Gamma_2)]^{\text{virt}}\right),$$

where

$$\Delta : D^\rho \to D^\rho \times D^\rho$$

is the diagonal morphism and m_ϵ is a rational number (multiplicity divided by the degree of Φ_ϵ).

5.3.2. *Decomposition of* $\mathfrak{M}(W_0,\Gamma)$. We will study the properties of $\mathfrak{M}(W_0,\Gamma)$ and their virtual fundamental classes in the setting of Section 3.1. A comprehensive comparison of the curve classes in X, Y and $\tilde{Y}$ is collected in the following diagram.

$$\begin{array}{ccccccccc}
H_3(M,\partial M) & \longrightarrow & H_2(\bigcup_i E_i) & \longrightarrow & H_2(\tilde{Y}) & \longrightarrow & H_2(M,\partial M) & \longrightarrow & 0 \\
\downarrow{=} & & \downarrow{\bar{\phi}_*} & & \downarrow{\phi_*} & & \downarrow{=} & & \downarrow{=} \\
H_3(M,\partial M) & \longrightarrow & H_2(\bigcup_i C_i) & \longrightarrow & H_2(Y) & \longrightarrow & H_2(M,\partial M) & \longrightarrow & 0 \\
\downarrow{=} & & \downarrow{\bar{\chi}_*} & & \downarrow{\chi_*} & & \downarrow{=} & & \downarrow{=} \\
H_3(M,\partial M) & \longrightarrow & 0 & \longrightarrow & H_2(X) & \longrightarrow & H_2(M,\partial M) & \longrightarrow & 0
\end{array}$$

It is easy to see that there is a unique lifting $\tilde{\gamma}$ of γ satisfying (3.4). From this and the degeneration analysis we have the following lemma.

Lemma 5.6.

$$[\overline{M}(Y,\gamma)]^{\text{virt}} \sim [\mathfrak{M}(\tilde{Y}, D; \tilde{\gamma})]^{\text{virt}},$$

where $\sim$ *stands for "homotopy equivalence".* [2] *They define the same GW invariants.*

Because of this lemma, we will sometimes *abuse the notation and identify* $[\mathfrak{M}(\tilde{Y}, D; \tilde{\gamma})]^{\text{virt}}$ *with* $[\overline{M}(Y,\gamma)]^{\text{virt}}$.

[2] If π can be extended to a family over $\mathbb{P}^1$, then the two cycles are rationally equivalent.

Lemma 5.7. *In the case of complex degeneration in Section 3.1, images of $\Phi_{\tilde{\gamma}}$ for different $\tilde{\gamma}$ are disjoint from each other.*

Proof. This follows from Li's study of the corresponding moduli stacks. In this special case of $\rho = 0$, for any element in $\mathfrak{M}(W_0, \Gamma)$ there is only one way to split it into two "relative maps" (with one of them being empty). We note that this is not true in general, when there are more than one way of splitting of the maps to the central fiber. □

As discussed before, given $\beta \neq 0$, if $\tilde{\gamma}$ and $\tilde{\gamma}'$ both satisfy (3.4), in particular they are non-exceptional for $\tilde{\psi} : \tilde{Y} \to \bar{X}$, we have

$$\tilde{\gamma} - \tilde{\gamma}' = \sum_i a_i(\ell_i - \ell_i'),$$

where ℓ_i and ℓ_i' are the $\tilde{\psi}$ exceptional curve classes (two rulings) in E_i. By Proposition 3.1, there are only finitely many nonzero a_i.

For each $\tilde{\gamma}$ above, there is a unique γ in Y, which is non-extremal for $\psi : Y \to \bar{X}$, such that $\tilde{\gamma}$ satisfies (3.6).

Corollary 5.8. *Given $\beta \neq 0$ a curve class in X, we can associate to it sets of non-$\tilde{\psi}$-exceptional curve classes $\tilde{\gamma}$ and γ discussed above. Then*

$$[\overline{M}(X, \beta)]^{\text{virt}} \sim \sum_{\tilde{\gamma}} [\mathfrak{M}(\tilde{Y}, D; \tilde{\gamma})]^{\text{virt}} \sim \sum_{\gamma} [\overline{M}(Y, \gamma)]^{\text{virt}},$$

where $\sim$ stands for the homotopy equivalence and the summations are over the above sets.

Notice that the conclusion holds for any projective small resolution Y of $\bar{X}$.

Proof. This follows from (3.3), (3.5) and the above discussions. □

Recall in Section 5.2 we have the identification of the linking data in

$$H_2(Y^\circ) = H_2(Y) = H_2(X^\circ) = H_2(X \setminus D) = H_2(\bar{X} \setminus \bar{X}^{\text{sing}}) \tag{5.4}$$

where

$$X \setminus \bigcup_{i=1}^{k} S_i =: X^\circ \sim M \sim Y^\circ := Y \setminus \bigcup_{i=1}^{k} C_i$$

and D is a tubular neighborhood of the union of vanishing S^3's

$$D = \bigcup_i D_i \cong \bigcup_i S^3 \times D^3.$$

Therefore, a curve class $\gamma \in H_2(Y)$ can be identified as a "curve class" in $X^\circ \sim \bar{X} \setminus \bar{X}^{\text{sing}}$, with the latter a quasi-projective variety. Therefore, we can think of γ as a curve class in X°.

Proposition 5.9. *For X_t with $t \in \mathbb{A}^1$ very small in the degenerating family*

$$\pi : \mathcal{X} \to \mathbb{A}^1,$$

we have a decomposition of the virtual class $[\overline{M}(X_t,\beta)]^{\text{virt}}$ into a finite disjoint union of cycles

$$[\overline{M}(X_t,\beta)]^{\text{virt}} = \coprod_{\gamma\in H_2(X^\circ)} [\overline{M}(X_t,\gamma)]^{\text{virt}},$$

where

$$[\overline{M}(Y,\gamma)]^{\text{virt}} \sim [\overline{M}(X_t,\gamma)]^{\text{virt}} \in A_{\text{vdim}}\left(\overline{M}(X_t,\beta)\right)$$

is a cycle class corresponding to the linking data γ of X_t.

Proof. By the construction of the virtual class of the family π, we know that the virtual classes for X_t and for X_0 are restrictions of that for $\mathcal{X}$. Lemma 5.7 tells us that at $t=0$, the virtual class decomposes into a disjoint union. By semicontinuity of connected components, we conclude that the virtual classes for X_t remain disconnected with (at least) the same number of connected components labeled by $\gamma \in H_2(X^\circ)$. □

We call the numbers defined by $[\overline{M}(X_t,\gamma)]^{\text{virt}}$ the *refined GW numbers* of X° with linking data γ.

Corollary 5.10. *The refined GW numbers of X° with linking data γ are the same as the GW invariants of Y with curve class γ, where γ is interpreted in two ways via* (5.4).

This corollary shows that $A(X) + B(X)_{\text{classical}} \Rightarrow A(Y)$.

Remark 5.11. According to Fukaya [8], if one allows the deformation of the almost complex structures J, the pseudo-holomorphic curves in a Calabi–Yau threefold do not intersect any number of given Lagrangian S^3 for generic J. Those J's for which some pseudo-holomorphic curves intersect some vanishing S^3 form a codimension 1 walls in the space of almost complex structures. These walls divide the space of almost complex structures into chambers. When one moves from one chamber to another, the wall crossing effect consists of counting pseudo-holomorphic disks with boundaries on the vanishing S^3. That is, the difference between counting of pseudo-holomorphic curves with J in one chamber and that with J in another is accountable by pseudo-holomorphic disk counting.

The results above, in particular Proposition 5.9 can be interpreted in the following way. *If* we know that the (*integrable*) complex structures in our moduli lie in (the interior of) the chambers, then the curve classes will never intersect the union of the vanishing S^3's. Therefore, the moduli of stable maps with a fixed β has a natural partition into disjoint unions of those with curve classes $\gamma \in H_2(X^\circ)$. Even though we do not know if this holds in general, Proposition 5.9 says that this still holds at the level of virtual classes when $[X]$ is sufficiently close to $[\bar{X}]$ in the moduli. Once we move far away from $[\bar{X}]$, the wall crossing is possible. Thus, the refined GW *numbers* for (X°,γ) are not symplectic invariants (with respect to X). In a work in progress [18], we plan to prove a blowup formula for genus zero which will cover any smooth blowups and some singular cases as well. That blowup will give $A(X) + B(X)_{\text{classical}} \Rightarrow A(Y)$, removing the constraint that $[X]$ must be sufficiently close to $[\bar{X}]$.

6. FROM $A(Y) + B(Y)$ TO $A(X) + B(X)$

6.1. **Overview.**

6.1.1. $A(Y) \Rightarrow A(X)$. As is explained in Section 3, $A(X)$ is a sub-theory of $A(Y)$. Indeed, $A(X)$ is obtained from $A(Y)$ by setting all extremal ray invariants to be zero, in addition to "reducing the linking data" $\gamma \in NE(Y)$ to $\beta \in NE(X)$.

6.1.2. $A(Y) + B(Y) \Rightarrow B(X)$. We have seen earlier that $B(Y)$ can be considered as a sub-theory of $B(X)$. In this section, we will show that $B(Y)$, together with the knowledge of extremal curves $\bigcup_i C_I \subset Y$ uniquely determines $B(X)$. More precisely, we will show that the "Hodge filtration" underlying the variation of MHS of the quasi-projective $Y^\circ = Y \setminus \bigcup_i C_i$ on the first jet space of $\mathcal{M}_Y \subset \mathcal{M}_{\bar{X}}$ can be lifted uniquely to the Hodge filtration underlying the degenerating VHS of X. Furthermore, the information of the Gauss–Manin up to the first jet is sufficient to uniquely single out the VHS of X.

In the next subsection, we start with a statement of compatibility of MHS which is needed in our discussion. After that we will give a proof showing the unique determination. As in our implication of $B(X) + A(X) \Rightarrow A(Y)$ in Section 5, our $A(Y) + B(Y) \Rightarrow B(X)$ implication is not constructive. It seems likely that a constructive recipe of this determination can be worked out by detailed analysis on the logarithmic model of degeneration of Hodge bundles by Steenbrink in [35] (see also [5]).

6.2. **Compatibility of the mixed Hodge structures.** Recall from Section 4.1 that $\mathcal{M}_{\bar{X}}$ is smooth and contains $\mathcal{M}_X$ as an open subscheme with "boundary" $\mathcal{M}_{\bar{X}} \setminus \mathcal{M}_X \cong \mathcal{M}_Y$. Set

$$U := Y^\circ = Y \setminus \bigcup_{i=1}^k C_i \cong \bar{X}^\circ = \bar{X} \setminus \bar{X}^{\text{sing}}$$

where $\bar{X}^{\text{sing}} = \bigcup_{i=1}^k \{p_i\}$.

To construct the VHS with logarithmic degeneration on $\mathcal{M}_{\bar{X}}$ near $\mathcal{M}_Y$, we start with the following lifting property.

Proposition 6.1. *There is a short exact sequence of mixed Hodge structures*

$$0 \to V \to H^3(X) \to H^3(U) \to 0, \tag{6.1}$$

where $H^3(X)$ is equipped with the limiting MHS of Schmid, $V \cong H^{1,1}_\infty H^3(X)$, and $H^3(U)$ is equipped with the canonical mixed Hodge structure of Deligne.

In particular, $F^3H^3(X) \cong F^3H^3(U)$ and $F^2H^3(X) \cong F^2H^3(U)$.

Proof. In the topological level, the short exact sequence (6.1) is equivalent to the defining sequence of the vanishing cycle space (2.10). Indeed, since X is nonsingular, $H_3(X) \cong H^3(X)$ by Poincaré duality. Also,

$$H_3(\bar{X}) = H_3(\bar{X}, p) \cong H_3(\tilde{Y}, E) \cong H^3(\tilde{Y} \backslash E) = H^3(U) \tag{6.2}$$

by the excision theorem and Lefschetz duality.

Now we consider the mixed Hodge structures. Since U is smooth quasi-projective, it is well know that the canonical mixed Hodge structure on $H^3(U)$ has its Hodge diamond supported on the upper triangular part, i.e., with weights ≥ 3. Or equivalently, the MHS on $H_3(\bar{X})$ has weights ≤ 3 by duality in (6.2). The crucial point is that Lefschetz duality is compatible with mixed Hodge structures, as stated in Lemma 6.2 below. Hence the short exact sequence (6.1) follows from Lemma 2.6 which is essentially the invariant cycle theorem.

Notice that $V \cong H^{1,1}_{\infty} H^3(X)$ by Lemma 2.6 (ii). In particular, the isomorphisms on F^i for $i = 3, 2$ follows immediately by applying F^i to (6.1). □

Lemma 6.2. *Let Y be an n dimensional complex projective variety, $i : Z \hookrightarrow Y$ a closed subvariety with smooth complement $j : U \hookrightarrow Y$ where $U := Y \backslash Z$. Then the Lefschetz duality*

$$H_i(Y, Z) \cong H^{2n-i}(U)$$

is compatible with the canonical mixed Hodge structures.

This is well known in mixed Hodge theory, though we are not able to locate an exact reference in the literature. For the readers' convenience we include a proof which is communicated to us by M. de Caltaldo.

Proof. We will make use of the structural theorem of Saito on mixed Hodge modules (MHM) [31, Theorem 0.1] which says that there is a correspondence between the derived categories of MHM and that of perverse sheaves (c.f. Axiom A in 14.1.1 of Peters and Steenbrink's book [28]).

There is a triangle in the derived category of constructible sheaves

$$j_! j^! \mathbb{Q}_Y \to \mathbb{Q}_Y \to i_* i^* \mathbb{Q}_Y.$$

This triangle gives maps of MHS:

$$H^i(Y, Z) \to H^i(Y) \to H^i(Z)$$

with

$$H^i(Y, Z) = H^i(Y, j_! j^! \mathbb{Q}_Y).$$

In fact, the MHS of $H^i(Y, Z)$ can be defined by the RHS from Saito's theory, since $j_! j^! \mathbb{Q}_Y$ is a complex of MHM.

Dualizing the above setup, we have

$$H_i(Y, Z) = H_i(Y, j_! j^! \mathbb{Q}_Y)^*, \tag{6.3}$$

where the LHS of (6.3) having MHS for the same reason as above and compatibly with taking dual as MHS. Furthermore, the RHS of (6.3) is

$H_c^{-i}(Y, j_*j^*\omega_Y)$ by Verdier duality, where ω_Y is the Verdier dualizing complex. Due to the compactness of Y we have

$$\begin{aligned} H_c^{-i}(Y, j_*j^*\omega_Y) &= H^{-i}(Y, j_*j^*\omega_Y) = H^{-i}(U, \omega_U) \\ &= H_i^{BM}(U) = H^{2n-i}(U), \end{aligned}$$

where H^{BM} is the Borel–Moore homology. Since every step above is compatible with MHM, it shows that the Lefschetz duality is compatible with the MHS. □

6.3. **Conclusion of the proof.** We now apply the above result to our setting. We have on $\bar{X}$ (cf. [27])

$$\cdots H^1_{\bar{X}^{\text{sing}}}(\Theta_{\bar{X}}) \to H^1(\Theta_{\bar{X}}) \to H^1(U, T_U) \to H^2_{\bar{X}^{\text{sing}}}(\Theta_{\bar{X}}) \to \cdots.$$

Since each p_i is a hypersurface singularity, we have depth $\mathcal{O}_{p_i} = 3$. Using this fact, Schlessinger showed that

$$H^1_p(\Theta_{\bar{X}}) = 0 \quad \text{and} \quad H^2_p(\Theta_{\bar{X}}) \cong \bigoplus_{i=1}^k \mathbb{C}_{p_i}.$$

Putting these together, we have

$$0 \to H^1(\Theta_{\bar{X}}) \to H^1(U, T_U) \to H^2_{\bar{X}^{\text{sing}}}(\Theta_{\bar{X}}) \to \cdots. \tag{6.4}$$

Since $\bar{X}$ is a Calabi–Yau 3-fold with only ODPs, its deformation theory is unobstructed by the T^1-lifting property [14]. Comparing (6.4) with (4.1) we see that

$$\text{Def}(\bar{X}) \cong H^1(U, T_U).$$

Similarly, on Y we have

$$\cdots H^1_Z(T_Y) \to H^1(T_Y) \to H^1(U, T_U) \to H^2_Z(T_Y) \to H^2(T_Y) \to \cdots.$$

Recall that Y is smooth Calabi–Yau and we have $H^1_Z(T_Y) = 0$. Thus

$$\text{Def}(Y) = H^1(T_Y) \subset H^1(U, T_U) \cong \text{Def}(\bar{X}).$$

$\mathcal{M}_Y$ is a natural submanifold of $\mathcal{M}_{\bar{X}}$. Write $\mathcal{I} := \mathcal{I}_{\mathcal{M}_Y}$ as the ideal sheaf of $\mathcal{M}_Y \subset \mathcal{M}_{\bar{X}}$.

Since $H^2(U, T_U) \neq 0$, the deformation of U could be obstructed. Nevertheless, the first-order deformation of U exists and is parameterized by $H^1(U, T_U) \supset \text{Def}(Y)$. Therefore, we have the following *smooth family*

$$\pi : \mathfrak{U} \to \mathcal{Z}_1 := Z_{\mathcal{M}_{\bar{X}}}(\mathcal{I}^2) \supset \mathcal{M}_Y,$$

where $\mathcal{Z}_1 = Z_{\mathcal{M}_{\bar{X}}}(\mathcal{I}^2)$ stands for the nonreduced subscheme of $\mathcal{M}_{\bar{X}}$ defined by the ideal sheaf $\mathcal{I}^2$. Namely $\mathcal{Z}_1$ is the first jet extension of $\mathcal{M}_Y$ in $\mathcal{M}_{\bar{X}}$.

Now we may complete the construction of VHS over $\mathcal{M}_{\bar{X}}$ near the boundary loci $\mathcal{M}_Y \hookrightarrow \mathcal{M}_{\bar{X}}$. The Gauss–Manin connection for a smooth family over non-reduced base was constructed in [13]. For our smooth family $\pi : \mathcal{U} \to \mathcal{Z}_1$, it is defined by the integral lattice $H^3(U, \mathbb{Z}) \subset H^3(U, \mathbb{C})$. Since U is only quasi-projective, the Gauss–Manin connection underlies

VMHS instead of VHS. By Proposition 6.1, we have $W_i H^3(U) = 0$ for $i \le 2$, $W_3 \subset W_4$ with $\mathrm{Gr}_3^W H^3(U) \cong H^3(Y)$, and $\mathrm{Gr}_4^W H^3(U) \cong V^*$.

The Hodge filtration of the locally system $F^0 = H^3(U, \mathbb{C})$ has the following structure: $F^\bullet = \{F^3 \subset F^2 \subset F^1 \subset F^0\}$ which satisfies the Griffiths transversality. Since $K_U \cong \mathcal{O}_U$ and $H^0(U, K_U) \cong H^0(Y, K_Y) \cong \mathbb{C}$, F^3 is a line bundle over $\mathcal{Z}_1$ spanned by a nowhere vanishing relative holomorphic 3-form $\Omega \in \Omega^3_{\mathcal{U}/\mathcal{Z}_1}$. Near the moduli point $[Y] \in \mathcal{Z}_1$, F^2 is then spanned by Ω and $v(\Omega)$ where v runs through a basis of $H^1(U, T_U)$. Notice that $v(\Omega) \in W_3$ precisely when $v \in H^1(Y, T_Y)$.

By Proposition 6.1, the partial Hodge filtration $F^3 \subset F^2$ on $H^3(U)$ over $\mathcal{Z}_1$ lifts uniquely to a filtration $\tilde{F}^3 \subset \tilde{F}^2$ on $H^3(X)$ over $\mathcal{Z}_1$ with $\tilde{F}^3 \cong F^3$ and $\tilde{F}^2 \cong F^2$. The complete lifting $\tilde{F}^\bullet$ is then uniquely determined since

$$\tilde{F}^1 = (\tilde{F}^3)^\perp$$

by the first Hodge–Riemann bilinear relation on $H^3(X)$. Alternatively, $\tilde{F}^1$ is spanned by $\tilde{F}^2$ and $v(\tilde{F}^2)$ for v runs through a basis of $H^1(U, T_U)$.

Now $\tilde{F}^\bullet$ over $\mathcal{Z}_1$ uniquely determines a horizontal map

$$\mathcal{Z}_1 \to \check{D}.$$

Since it has maximal tangent dimension $h^1(U, T_U) = h^1(X, T_X)$, it determines uniquely the maximal horizontal slice

$$\psi : \mathcal{M} \to \check{D}$$

with $\mathcal{M} \cong \mathcal{M}_{\bar{X}}$ locally near $\mathcal{M}_Y$. According to Theorem 4.15, namely an extension of Schmid's nilpotent orbit theorem, under the coordinates $\mathbf{t} = (r, s)$, the period map

$$\phi : \mathcal{Z}^\times \cong \mathcal{M}_X = \mathcal{M}_{\bar{X}} \backslash \bigcup\nolimits_{i=1}^k D^i \to D/\Gamma$$

is then given by

$$\phi(r, s) = \exp\left(\sum_{i=1}^k \frac{\log w_i}{2\pi\sqrt{-1}} N^{(i)}\right) \psi(r, s)$$

where Γ is the monodromy group generated by the local monodromy $N^{(i)}$ around the divisor D^i defined by $w_i = \sum_{j=1}^\mu a_{ij} r_j = 0$ (c.f. (4.7)). Since $N^{(i)}$ is determined by the Picard–Lefschetz formula (Lemma 4.14), we see that the period map ϕ is completely determined by the relation matrix A of the extremal curves C_i's. (The period map gives the desired VHS, with degenerations, over $\mathcal{Z}^\times$.) This completes the proof that refined B model on $Y \backslash Z = U$ determines the B model on X.

Remark 6.3. Bryant and Griffiths reformulate the VHS for Calabi–Yau threefolds in terms of Legendre subvarieties in $P(H^3(X))$ It might be possible to show that $\tilde{F}^\bullet$ over $\mathcal{Z}_1$ uniquely determines a Legendre subvariety inside $P(H^3(X))$ which coincides with $\mathcal{M}_{\bar{X}}$.

7. Remarks on the basic exact sequence

For a conifold transition of Calabi–Yau 3-folds $X \nearrow Y$, we have shown that the combined information of A model and B model of X determines the corresponding information on Y, and vice versa. However, the effective computational method for such a determination has not been addressed much besides the vanishing/extremal invariants.

In this final section of the paper we make two remarks concerning the quantum aspects of the basic exact sequence. The aim is to shed some light on the possible directions to achieve such an effective computational method, especially on the implication $A(X) + B(X) \Rightarrow A(Y)$.

7.1. **Local transitions between** $A(Y)$ **and** $B(X)$**.** The basic exact sequence in Theorem 2.9 provides a Hodge theoretic realization of the numerical identity $\mu + \rho = k$. Now $H^2(Y)/H^2(X) \otimes \mathbb{C} \cong \mathbb{C}^\rho$ is naturally the parameter space of the extremal Gromov–Witten invariants of the Kähler degeneration $\psi : Y \to \bar{X}$, and $V^* \otimes \mathbb{C} \cong \mathbb{C}^\mu$ is naturally the parameter space of periods of vanishing cycles of the complex degeneration from X to $\bar{X}$. Both of them are equipped with flat connections induced from the Dubrovin (resp. Gauss–Manin) connection over their tangent bundles. Thus it is natural to ask if there is a $\mathcal{D}$ module lift of the basic exact sequence.

We rewrite the basic exact sequence in the form

$$\begin{array}{ccc} & \mathbb{C}^k & \\ {\scriptstyle B}\nearrow & & \nwarrow{\scriptstyle A} \\ H^2_{\mathbb{C}}(Y)/H^2_{\mathbb{C}}(X) \cong \mathbb{C}^\rho & & V^*_{\mathbb{C}} \cong \mathbb{C}^\mu \end{array}$$

with $A^t B = 0$. This simply means that $\mathbb{C}^k$ is an orthogonal direct sum of the two subspaces $\operatorname{im}(A)$ and $\operatorname{im}(B)$. Let $A = [A^1, \cdots, A^\mu]$, $B = [B^1, \cdots, B^\rho]$, and consider the invertible matrix

$$S = (s^i_j) := [A, B] \in M_{k\times k}(\mathbb{Z}),$$

namely $s^i_j = a_{ij}$ for $1 \le j \le \mu$ and $s^i_{\mu+j} = b_{ij}$ for $1 \le j \le \rho$.

Denote the standard basis of $\mathbb{C}^k$ by $e_1, \cdots, e_k$ with coordinates $y_1, \cdots, y_k$. Let $e^1, \cdots, e^k$ be the dual basis on $(\mathbb{C}^k)^\vee$. We consider the standard (trivial) logarithmic connection on the bundle $\underline{\mathbb{C}}^k \oplus (\underline{\mathbb{C}}^k)^\vee$ over $\mathbb{C}^k$ defined by

$$\nabla = d + \frac{1}{z} \sum_{i=1}^k \frac{dy_i}{y_i} \otimes (e^i \otimes e_i^*), \tag{7.1}$$

where z is a parameter. It is a direct sum of k copies of its one dimensional version. We will show that the principal (logarithmic) part of the Dubrovin connection over $\mathbb{C}^\rho$ (c.f. (3.8)) as well as the Gauss–Manin connection on $\mathbb{C}^\mu$ (c.f. (4.11)) are all induced from this standard logarithmic connection through the embeddings defined by B and A respectively.

Recall the basis $T_1, \cdots, T_\rho$ of $\mathbb{C}^\rho$ with coordinates $u^1, \cdots, u^\rho$, and the frame $T_1, \cdots, T_\rho, T^1, \cdots, T^\rho$ on the bundle $\underline{\mathbb{C}}^\rho \oplus (\underline{\mathbb{C}}^\rho)^\vee$ over $\mathbb{C}^\rho$. Notice that T_j corresponds to the column vector $B^j = S^{\mu+j}$, $1 \le j \le \rho$. Let $\hat{T}_j$ correspond to the column vector $A^j = S^j$ for $1 \le j \le \mu$ with dual $\hat{T}^j$'s. Then

$$T_j = \sum_{i=1}^k b_{ij}\, e_i = \sum_{i=1}^k s^i_{\mu+j}\, e_i,$$

and dually

$$e^i = \sum_{j=1}^\mu s^i_j\, \hat{T}^j + \sum_{j=1}^\rho s^i_{\mu+j}\, T^j = \sum_{j=1}^\mu a_{ij}\, \hat{T}^j + \sum_{j=1}^\rho b_{ij}\, T^j. \tag{7.2}$$

Denote by P the orthogonal projection

$$P : \underline{\mathbb{C}}^k \oplus (\underline{\mathbb{C}}^k)^\vee \to \underline{\mathbb{C}}^\rho \oplus (\underline{\mathbb{C}}^\rho)^\vee.$$

Using (7.1) and (7.2), we compute the induced connection ∇^P near $\vec{0} \in \mathbb{C}^\rho$:

$$\begin{aligned} \nabla^P_{T_l} T_m &= \sum_{i,\, i'=1}^k b_{il} b_{i'm} \big(\nabla_{e_i} e_{i'}\big)^P \\ &= \frac{1}{z} \sum_{i=1}^k \frac{b_{il} b_{im}}{y_i}\, (e^i)^P = \frac{1}{z} \sum_{n=1}^\rho \sum_{i=1}^k \frac{b_{il} b_{im} b_{in}}{y_i}\, T^n. \end{aligned} \tag{7.3}$$

We compare it with the one obtained in (3.8), (3.9) and (3.11):

$$\nabla^z_{T_l} T_m = -\frac{1}{z} \sum_{n=1}^\rho \left((T_l.T_m.T_n) + \sum_{i=1}^k b_{il} b_{im} b_{in} \frac{q_i}{1-q_i} \right) T^n,$$

where $q_i = \exp \sum_{p=1}^\rho b_{ip} u^p = \exp v_i$. Thus the principal part near $u_i = 0$, $1 \le i \le \rho$, gives

$$\frac{1}{z} \sum_{n=1}^\rho \sum_{i=1}^k \frac{b_{il} b_{im} b_{in}}{v_i}\, T^n,$$

which coincides with (7.3) by setting $v_i = y_i$ for $1 \le i \le \rho$. We summarize the discussion in the following proposition:

Proposition 7.1. *Let $X \nearrow Y$ be a projective conifold transition through $\bar{X}$ with k ordinary double points. Let the bundle $\underline{\mathbb{C}}^k \oplus (\underline{\mathbb{C}}^k)^\vee$ over $\mathbb{C}^k$ be equipped with the standard logarithmic connection defined in* (7.1). *Then*

(1) *The connection induced from the embedding $B : \mathbb{C}^\rho \to \mathbb{C}^k$ defined by the relation matrix of vanishing 3 spheres for the degeneration from X to $\bar{X}$ gives rise to the logarithmic part of the Dubrovin connection on $H^2(Y)/H^2(X)$.*

(2) *The connection induced from the embedding $A : \mathbb{C}^\mu \to \mathbb{C}^k$ defined by the relation matrix of extremal rational curves for the small contraction $Y \to \bar{X}$ gives rise to the logarithmic part of the Gauss–Manin connection on V^*, where V is the space of vanishing 3-cycles.*

Part (1) has just been proved. The proof for (2) is similar (by setting $z = 2\pi\sqrt{-1}$ and $w_i = y_i$, c.f. (4.11)) and is omitted. We remark that the two subspaces $B(\mathbb{C}^\rho)$ and $A(\mathbb{C}^\mu)$ are indeed defined over $\mathbb{Q}$ and orthogonal to each other, hence A and B determine each other up to choices of basis.

7.2. **Speculation for globalization.** Our proof for $A(X) + B(X) \Rightarrow A(Y)$ in Section 5.3 is not constructive. Here we discuss briefly an idea developed in a forthcoming work to attack the problem for genus zero theory [18].

We have seen that the Dubrovin connection on $H^2(Y)/H^2(X)$ is determined by the relation matrix B of vanishing spheres. Consider the diagram

$$\begin{array}{ccc} H^2(Y)/H^2(X) & \longleftarrow & H^2(Y) \\ & & \uparrow \\ & & H^2(X), \end{array}$$

and regard it as the cohomology realization of the small contraction

$$\begin{array}{ccc} \bigcup_{i=1}^k C_i & \longrightarrow & Y \\ & & \big\downarrow \bar{\psi} \\ & & \bar{X}. \end{array}$$

Since $\bar{X}$ is singular and not an orbifold, the Gromov–Witten theory on $\bar{X}$ is so far undefined in the literature. Nevertheless, in the current situation, according to the principle of deformation invariance we may treat it as $GW(X)$, which is given. Now the picture looks very similar to the quantum Leray–Hirsch theorem for projective (or toric) bundles proved in [17] despite the fact that $\bar{\psi}$ is a birational contraction/crepant blowup instead of a bundle morphism. However, in the cohomology level it looks just like a bundle. Thus it is reasonable to believe that the idea of quantum Leray–Hirsch principle can also be applied to such a situation.

To see how the B model of X enters the picture, we mention only the observation that $\bar{\psi} : Y \to \bar{X}$ can be realized as the blow-up along certain Weil divisors in $\bar{X}$, and those Weil divisors can in fact be constructed from the relation matrix B of the k vanishing spheres S_i's.

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Y.-P. LEE: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF UTAH, SALT LAKE CITY, UTAH 84112-0090, U.S.A.
E-mail address: yplee@math.utah.edu

H.-W. LIN: DEPARTMENT OF MATHEMATICS AND TAIDA INSTITUTE FOR MATHEMATICAL SCIENCES (TIMS), NATIONAL TAIWAN UNIVERSITY, TAIPEI 10617, TAIWAN
E-mail address: linhw@math.ntu.edu.tw

C.-L. WANG: DEPARTMENT OF MATHEMATICS, CENTER FOR ADVANCED STUDIES IN THEORETICAL SCIENCES (CASTS), AND TAIDA INSTITUTE FOR MATHEMATICAL SCIENCES (TIMS), NATIONAL TAIWAN UNIVERSITY, TAIPEI 10617, TAIWAN
E-mail address: dragon@math.ntu.edu.tw