THE 5-CANONICAL SYSTEM ON 3-FOLDS OF GENERAL TYPE

JUNGKAI A. CHEN, MENG CHEN AND DE-QI ZHANG

Abstract. Let X be a projective minimal Gorenstein 3-fold of general type with canonical singularities. We prove that the 5-canonical map is birational onto its image.

1. Introduction

One main goal of algebraic geometry is to classify algebraic varieties. The successful 3-dimensional MMP (see [16, 19] for example) has been attracting more and more mathematicians to the study of algebraic 3-folds. In this paper, we restrict our interest to projective minimal Gorenstein 3-folds X of general type where there still remain many open problems.

Denote by K_X the canonical divisor and $\Phi_m := \Phi_{|mK_X|}$ the m-canonical map. There have been a lot of works along the line of the canonical classification. For instance, when X is a smooth 3-fold of general type with pluri-genus $h^0(X, kK_X) \geq 2$, in [17], as an application to his research on higher direct images of dualizing sheaves, Kollár proved that Φ_m, with $m = 11k + 5$, is birational onto its image. This result was improved by the second author [5] to include the cases m with $m \geq 5k + 6$; see also [7], [9] for results when some additional restrictions (like bigger $p_g(X)$) were imposed.

On the other hand, for 3-folds X of general type with $q(X) > 0$, Kollár [17] first proved that Φ_{225} is birational. Recently, the first author and Hacon [4] proved that Φ_m is birational for $m \geq 7$ by using the Fourier-Mukai transform. Moreover, Luo [22], [23] has some results for 3-folds of general type with $h^2(\mathcal{O}_X) > 0$.

Now for minimal and smooth projective 3-folds, it has been established that Φ_m ($m \geq 6$) is a birational morphism onto its image after 20 years of research, by Wilson [29] in the year 1980, Benveniste [2] in the year 1986 ($m \geq 8$), Matsuki [24] in the year 1986 ($m = 7$), the second author [6] in the year 1998 ($m = 6$) and independently by Lee

The first author was partially supported by the National Science Council and National Center for Theoretical Science of Taiwan. The second author was supported by the National Natural Science Foundation of China. The third author is supported by an Academic Research Fund of NUS.

[20], [21] in the years 1999-2000 ($m = 6$; and also the base point freeness of m-canonical system for $m \geq 4$). A very natural and well-known question arises:

Question 1.1. Let X be a minimal Gorenstein 3-fold of general type. Is Φ_5 birational onto its images?

Despite many attempts officially or privately announced, it seems that the birationality of Φ_5 for 3-folds (even with the stronger assumption that K_X is ample) remains beyond reach. The difficulty lies in the case with smaller $p_g(X)$ or K_X^3. One reason to account for this is that the non-birationality of the 4-canonical system for surfaces may happen when they have smaller p_g or K^2 (see Bombieri [3]), whence a naive induction on the dimension would predict the non-birationality of the 5-canonical system on certain 3-folds with smaller invariants.

Nevertheless, there is also evidence supporting the birationality of Φ_5 for Gorenstein minimal 3-folds X of general type. For instance, one sees that $K_X^3 \geq 2$ for minimal and smooth X (see 2.1 below). So an analogy of Fujita's conjecture would predict that $|5K_X|$ gives a birational map. We recall that Fujita's conjecture (the freeness part) has been proved by Fujita, Ein-Lazarsfeld [10] and Kawamta [14] when $\dim X \leq 4$.

The aim of this paper is to answer Question 1.1 which has been around for many years:

Theorem 1.2. *Let X be a projective minimal Gorenstein 3-fold of general type with canonical singularities. Then the m-canonical map Φ_m is a birational morphism onto its image for all $m \geq 5$.*

Example 1.3. The numerical bound "5" in Theorem 1.2 is optimal. There are plenty of supporting examples. For instance, let $f : V \longrightarrow B$ be any fibration where V is a smooth projective 3-fold of general type and B a smooth curve. Assume that a general fiber of f has the minimal model S with $K_S^2 = 1$ and $p_g(S) = 2$. (For example, take the product.) Then $\Phi_{|4K_V|}$ is apparently not birational (see [3]).

1.4. Reduction to birationality. According to [6] or [20], to prove Theorem 1.2, we only need to verify the statement in the case $m = 5$. On the other hand, the results in [20, 21] show that $|mK_X|$ is base point free for $m \geq 4$. So it is sufficient for us to verify the birationality of $|5K_X|$ in this paper.

1.5. Reduction to factorial models. According to the work of M. Reid [26] and Y. Kawamata [15] (Lemma 5.1), there is a minimal model Y with a birational morphism $\nu : Y \longrightarrow X$ such that $K_Y = \nu^*(K_X)$ and that Y is factorial with at worst terminal singularities. Thus it is sufficient for us to prove Theorem 1.2 for minimal factorial models.

Acknowledgments. We are indebted to Hélène Esnault, Christopher Hacon, Yujiro Kawamata, Miles Reid, I-Hsun Tsai, Eckart Viehweg and Chin-Lung Wang for useful conversations or comments on this subject.

2. Notation, Formulae and Set up

We work over the complex number field $\mathbb{C}$. By *a minimal variety* X, we mean one with nef K_X and with terminal singularities (except when we specify the singularity type).

2.1. Let X be a projective minimal Gorenstein 3-fold of general type. Taking a special resolution $\nu : Y \longrightarrow X$ according to Reid ([26]) such that $c_2(Y)\cdot\triangle = 0$ (see Lemma 8.3 of [25]) for any exceptional divisor $\triangle$ of ν. Write $K_Y = \nu^* K_X + E$ where E is exceptional and is mapped to a finite number of points. Then for $m \geq 2$, we have (by the vanishing in [13], [28] or [11]):

$$\chi(\mathcal{O}_X) = \chi(\mathcal{O}_Y) = -\frac{1}{24}K_Y \cdot c_2(Y) = -\frac{1}{24}\nu^* K_X \cdot c_2(Y).$$

$$\begin{aligned} P_m(X) &= \chi(\mathcal{O}_X(mK_X)) = \chi(\mathcal{O}_Y(m\nu^* K_X)) \\ &= \frac{1}{12}m(m-1)(2m-1)K_X^3 + \frac{m}{12}\nu^* K_X \cdot c_2(Y) + \chi(\mathcal{O}_Y) \\ &= (2m-1)(\frac{m(m-1)}{12}K_X^3 - \chi(\mathcal{O}_X)). \end{aligned}$$

The inequality of Miyaoka and Yau ([25], [30]) says that $3c_2(Y) - K_Y^2$ is pseudo-effective. This gives $\nu^* K_X \cdot (3c_2(Y) - K_Y^2) \geq 0$. Noting that $\nu^* K_X \cdot E^2 = 0$ under this situation, we get:

$$-72\chi(\mathcal{O}_X) - K_X^3 \geq 0.$$

In particular, $\chi(\mathcal{O}_X) < 0$. So one has:

$$q(X) = h^2(\mathcal{O}_X) + (1 - p_g(X)) - \chi(\mathcal{O}_X) > 0$$

whenever $p_g(X) \leq 1$.

2.2. Suppose that D is any divisor on a smooth 3-fold V. The Riemann-Roch theorem gives:

$$\chi(\mathcal{O}_V(D)) = \frac{D^3}{6} - \frac{K_V \cdot D^2}{4} + \frac{D \cdot (K_V^2 + c_2)}{12} + \chi(\mathcal{O}_V).$$

Direct calculation shows that

$$\chi(\mathcal{O}_V(D)) + \chi(\mathcal{O}_V(-D)) = \frac{-K_V \cdot D^2}{2} + 2\chi(\mathcal{O}_V) \in \mathbb{Z}.$$

Therefore, $K_V \cdot D^2$ is an even number.

Now let X be a projective minimal Gorenstein 3-fold of general type. Let D be any divisor on X. Then $K_X \cdot D^2 = K_Y \cdot (\nu^* D)^2$ is even. Especially K_X^3 is even and positive.

2.3. Let V be a smooth projective 3-fold and let $f : V \longrightarrow B$ be a fibration onto a nonsingular curve B. From the spectral sequence:

$$E_2^{p,q} := H^p(B, R^q f_*\omega_V) \Longrightarrow E^n := H^n(V, \omega_V),$$

one has the following by Serre duality and Corollary 3.2 and Proposition 7.6 on pages 186 and 36 of [17]:

$$h^2(\mathcal{O}_V) = h^1(B, f_*\omega_V) + h^0(B, R^1 f_*\omega_V),$$

$$q(V) := h^1(\mathcal{O}_V) = g(B) + h^1(B, R^1 f_*\omega_V).$$

2.4. For $\mu = 1, 2$, we set

$$\Phi = \begin{cases} \Phi_{|K_X|} & \text{if } p_g(X) \geq 2, \\ \Phi_{|2K_X|} & \text{otherwise.} \end{cases}$$

Since we always have $P_2(X) \geq 4$, Φ is a non-trivial rational map.

Let $\pi : X' \longrightarrow X$ be the a resolution of the base locus of Φ. We write $|\pi^*(\mu K_X)| = |M'| + E'$. Then we may assume:

(1) X' is smooth;

(2) the movable part of $|\mu K_{X'}|$ is $|M'|$, which is base point free;

(3) E' is a normal crossing divisor (hence so is a general member in $|\pi^*(\mu K_X)|$).

We will fix some notation below. The frequently used ones are M, Z, S, Δ and E_π. Denote by g the composition $\Phi \circ \pi$. So $g : X' \longrightarrow W' \subseteq \mathbb{P}^N$ is a morphism. Let $g : X' \xrightarrow{f} W \xrightarrow{s} W'$ be the Stein factorization of g such that W is normal and f has connected fibers. We can write:

$$|\mu K_{X'}| = |\pi^*(\mu K_X)| + \mu E_\pi = |M'| + Z',$$

where Z' is the fixed part and E_π an effective π-exceptional divisors.

On X, one may write $\mu K_X \sim M + Z$ where M is a general member of the movable part and Z the fixed divisor. Let $S \in |M'|$ be the divisor corresponding to M, then

$$\pi^*(M) = S + \triangle = S + \sum_{i=1}^{s} d_i E_i$$

with $d_i > 0$ for all i. The above sum runs over all those exceptional divisors of π that lie over the base locus of M. Obviously $E' = \triangle + \pi^*(Z)$. On the other hand, one may write $E_\pi = \sum_{j=1}^{t} e_j E_j$ where the sum runs over *all* exceptional divisors of π. One has $e_j > 0$ for all $1 \leq j \leq t$ because X is terminal. Apparently, one has $t \geq s$.

Note that $\mathrm{Sing}(X)$ is a finite set (see [19], Corollary 5.18). We may write $E_\pi = \triangle' + \triangle''$ where $\triangle'$ (resp. $\triangle''$) lies (resp. does not lie) over the base locus of $|M|$. So if one only requires such a modification π that satisfies 2.4(1) and 2.4(2), one surely has $\mathrm{supp}(\triangle) = \mathrm{supp}(\triangle')$.

Let $d := \dim \Phi(X)$. And let $L := \pi^*(K_X)_{|S}$, which is clearly nef and big. Then we have the following:

Lemma 2.5. *When $d \geq 2$, $(L^2)^2 \geq (\pi^* K_X)^3 (\pi^*(K_X) \cdot S^2)$. Morover, $L^2 \geq 2$.*

Proof. Take a sufficiently large number m such that $|m\pi^*(K_X)|$ is base point free. Denote by H a general member of this linear system. Then H must be a smooth projective surface. On H, we have nef divisors $\pi^*(K_X)_{|H}$ and $S_{|H}$. Applying the Hodge index theorem, one has

$$(\pi^*(K_X)_{|H} \cdot S_{|H})^2 \geq (\pi^*(K_X)_{|H})^2 (S_{|H})^2.$$

Removing m, we get the first inequality. By 2.2, $(\pi^* K_X)^3$ is even, hence ≥ 2. Together with $\pi^*(K_X) \cdot S^2 > 0$, we have the second inequality. □

We now state a lemma which will be needed in our proof. The result might be true for all 3-folds with rational singularities.

Lemma 2.6. *Let X be a normal projective 3-fold with only canonical singularities. Let M be a Cartier divisor on X. Assume that $|M|$ is a movable pencil and that $|M|$ has base points. Then $|M|$ is composed with a rational pencil.*

Proof. Take a birational morphism $\pi : X' \longrightarrow X$ such that X' is smooth, that the exceptional divisor E_π is of simple normal crossing, and that the map $\Phi_{|M|}$ composed with π, becomes a morphism from X' to a curve. Take a Stein factorization of the latter morphism to get an induced fibration $f : X' \longrightarrow B$ onto a smooth curve B. The lemma asserts that B must be rational.

Clearly, the exceptional divisor E_π dominates B.

Case 1. $Bs|M|$ contains a curve Γ.

This is the easier case. Note that X has only finitely many points at which K_X is non-Cartier or X is non-cDV (see Cor. 5.40 of [19]). So we can pick up a very ample divisor H on X (avoiding these finitely many points) such that H is Du Val and intersects Γ transversally. We may assume that the strict transform H' on X' is smooth, i.e., π is an embedded resolution of $H \subset X$. Clearly, there is an π-exceptional irreducible divisor E which dominates both Γ and B. Now for a general H, both H' and $E \cap H'$ dominate B. Since the curve $E \cap H'$ arises from the resolution $\pi : H' \to H$ of the indeterminancy of the linear system $|M|_{|H}$ (whose image on X is contained in $\Gamma \cap H$), it is rational. So B is rational.

Case 2. $Bs|M|$ is a finite set. (The argument below works even when X is log terminal.)

Take a base point P of $|M|$. Then $E = \pi^{-1}(P)$ dominates B, i.e., $f(E) = B$. By Kollar's Theorem 7.6 in [18], there is an analytic contractible neighborhood V of P such that $U = \pi^{-1}(V) \subset X'$ is simply connected. Suppose $g(B) > 0$. Then the universal cover $h : W \longrightarrow B$ of B is either the affine line $\mathbb{C}$ or an open disk in $\mathbb{C}$. By Proposition 13.5 of [12], there is a factorization for the restriction $f|_U : U \longrightarrow B$, say $f = h \circ m$, where $m : U \longrightarrow W$ is continuous. Note that $m(E)$ is a compact subset of W, so $m(E)$ is a single point. In particular, $f(E)$ is a point, a contradiction. □

3. The case $p_g \geq 2$

The following proposition is quite useful throughout the paper.

Proposition 3.1. *Let S be a smooth projective surface. Let C be a smooth curve on S, $N' < N$ be divisors on S and $\Lambda \subset |N|$ be a subsystem. Suppose that $|N'||_C = |N'_{|C}|$, $\deg(N_{|C}) = 1 + \deg(N'_{|C}) \geq 1 + 2g(C)$. We consider the following diagram*

$$\begin{array}{ccc} |N'| & \xrightarrow{res} & |N'_{|C}| \\ \big\downarrow {\scriptstyle +eff} & & \big\downarrow {\scriptstyle +P_1} \\ |N| & \xrightarrow{res} & |N_{|C}| \\ \big\uparrow {\scriptstyle \subset} & & \big\uparrow {\scriptstyle \subset} \\ \Lambda & \xrightarrow{res} & \Lambda_C \end{array}$$

Suppose furthermore that $\Lambda_{|C}$ is free and $\Lambda_{|C} \supset |N'||_C + P_1$. Then

$$\Lambda_{|C} = |N||_C = |N_{|C}|, \qquad (*)$$

which is very ample and complete.

Proof. By the Riemann-Roch theorem and Serre duality, we have dim $|N_{|C}| = 1 + \dim |N'_{|C}|$. Since there are inclusions $|N'||_C + P_1 \subseteq \Lambda_{|C} \subseteq |N||_C \subseteq |N_{|C}|$, now the equalities (*) in the statement follow from the dimension counting and the fact that the first inclusion above is strict by the freeness of $\Lambda_{|C}$. □

Theorem 3.2. *Let X be a projective minimal factorial 3-fold of general type. Assume $p_g(X) \geq 2$. Then Φ_5 is birational.*

Proof. We distinguish cases according to $d := \dim \Phi(X)$.

Case 1: $d = 3$. Then $p_g(X) \geq 4$. Φ_5 is birational, thanks to Theorem 3.1(i) in [9].

Case 2: $d = 2$. We consider the linear system $|K_{X'} + 3\pi^*(K_X) + S|$. Since $K_{X'} + 3\pi^*(K_X) + S \geq S$ and according to Tankeev's principle, it is sufficient to verify the birationality of $\Phi_{|K_{X'}+3\pi^*(K_X)+S|_{|S}}$. Note that we have a fibration $f : X' \longrightarrow W$ where a general fiber of f is a smooth curve C of genus ≥ 2. The vanishing theorem gives:

$$|K_{X'} + 3\pi^*(K_X) + S||_S = |K_S + 3L|$$

where $L := \pi^*(K_X)_{|S}$ is a nef and big divisor on S.

By Lemma 2.5, $L^2 \geq 2$. According to Reider ([27]), $\Phi_{|K_S+3L|}$ is birational and so is Φ_5.

Case 3: $d = 1$. We set $b := g(B)$. When $b > 0$, let's consider the system $|M|$ on X. If $|M|$ has base point, then by 2.6, $b = 0$, which is a contradiction. Thus we may assume that $|M|$ is free. Then in this situation, Φ_5 is birational, which is exactly the statement of Theorem 3.3 in [9]. For reader's convenience, we sketch the proof here. Meng, can you do this?

From now on, we suppose $b = 0$. Let F be a general fiber of f and denote by $\sigma : F \longrightarrow F_0$ the contraction onto the minimal model. We take π to be the composition $\pi_1 \circ \pi_0$ where π_0 satisfies 2.4(1) and 2.4(2) and π_1 is a further modification such that $\pi^*(K_X)$ is supported on a normal crossing divisor.

We may write $S \sim aF$ where $a \geq p_g(X) - 1$. And we set $L := \pi^*(K_X)_{|F}$ instead. From the relation

$$|K_{X'} + 3\pi^*(K_X) + S|_{|F} = |K_F + 3L|,$$

we see that the problem is reduced to the birationality of $|K_F + 3L|$ because $|K_{X'} + 3\pi^*(K_X) + S| \supset |S|$ apparently separates different fibers of f. Let $\bar{F} := \pi_*(F)$. We know that $K_X \cdot \bar{F}^2$ is an even number by 2.2.

If $K_X \cdot \bar{F}^2 > 0$, then we have

$$L^2 = \pi^*(K_X)^2 \cdot F = K_X^2 \cdot \bar{F} \geq K_X \cdot \bar{F}^2 \geq 2.$$

Reider's theorem says that $|K_F + 3L|$ gives a birational map.

We are left with only the case $K_X \cdot \bar{F}^2 = 0$. First we have:

Claim 3.3. *If $K_X \cdot \bar{F}^2 = 0$, then $\mathcal{O}_F(\pi^*(K_X)_{|F}) \cong \mathcal{O}_F(\sigma^* K_{F_0})$.*

Proof. It is obvious that the claim is true if it holds for $\pi = \pi_0$. So we may assume $\pi = \pi_0$. Now

$$0 = K_X \cdot (a\bar{F})^2 = K_X \cdot M^2 = \pi^*(K_X) \cdot \pi^*(M) \cdot S = a\pi^*(K_X)_{|F} \cdot \triangle_{|F},$$

which means $\pi^*(K_X)_{|F} \cdot \triangle'_{|F} = 0$. On the other hand, the definition of π_0 gives $\triangle''_{|F} = 0$. Thus $(E_\pi)_{|F} \cdot \pi^*(K_X)_{|F} = 0$.

We may write

$$K_F = \pi^*(K_X)_{|F} + G$$

where $G = (E_\pi)_{|F}$ is an effective and contractible (so negative definite) divisor on F. Note that L is nef and big and that $L \cdot G = 0$. The uniqueness of the Zariski decomposition shows that $\sigma^* K_{F_0} \sim \pi^*(K_X)_{|F}$. We are done. □

From the above claim, we have $\Phi_{|K_F+3L|} = \Phi_{|4K_F|}$. We are left to verify the birationality of Φ_5 only when $\Phi_{|4K_F|}$ fails to be birational, i.e. when $K_{F_0}^2 = 1$ and $p_g(F) = 2$.

The Kawamata-Viehweg vanishing theorem ([11, 13, 28]) gives

$$|K_{X'} + 3\pi^*(K_X) + F|_{|F} = |K_F + 3\sigma^*(K_{F_0})|. \quad (1)$$

Denote by C a general member of the movable part of $|\sigma^* K_{F_0}|$. By [1], we know that C is a smooth curve of genus 2 and $\sigma(C)$ is a general member of $|K_{F_0}|$. Applying the vanishing theorem again, we have

$$|K_F + 2\sigma^*(K_{F_0}) + C|_{|C} = |K_C + 2\sigma^*(K_{F_0})_{|C}|. \quad (2)$$

Now we may apply Proposition 3.1. Let $N' := K_F + 2\sigma^*(K_{F_0}) + C$ and $N := (5\pi^* K_X)_{|F}$. Set $\Lambda = |5\pi^*(K_X)|_{|F}$. It's clear that $N' < N$. Also note that Λ is free for $|5K_X|$ is free.

By (1) above, we see that $\Lambda \supset |N'|+$ (a fixed effective divisor).

Now restrict to C, computation shows that $\deg(N'_{|C}) = 4$ and $5 = \deg(N_{|C}) = 1 + \deg(N'_{|C})$. Therefore, the induced inclusion $|N'_{|C}| \hookrightarrow |N_{|C}|$ is given by adding a single point P_1.

By (2), we have $|N'_{|C}| = |N'|_{|C}$. Together with (1), we have $\Lambda_{|C} \supset |N'_{|C}| + P_1$. Hence by Proposition 3.1, $\Lambda_{|C} = |N_{|C}|$ gives an embedding. Because $|5\pi^* K_X|_{|F} \supset |N'| \supset |C|$ (by (1) above) separates different C (noting that $p_g(F) = 2$ and $|C|$ is a rational pencil), $\Phi_{5|F}$ is birational. It is clear that $|5\pi^* K_X| \supset |S|$ separates different fibres F. Thus Φ_5 is birational. □

4. **Birationality via bicanonical systems**

In this section, we shall complete the proof of Theorem 1.2 by studying the bicanonical system. We set $\Phi := \Phi_2$ as stated in 2.4. Denote $d_2 := \dim \Phi_2(X)$. We organize our proof according to the value of d_2.

Theorem 4.1. *Let X be a projective minimal factorial 3-fold of general type. Assume $d_2 = 3$. Then Φ_5 is birational.*

Proof. Recall that K_X^3 is even by 2.2, so it's either > 2 or $= 2$.

Case 1. The case $K_X^3 > 2$.

Pick up a general member S. Let $R := S_{|S}$. Then $|R|$ is not composed of a pencil. Thus one obviously has $R^2 \geq 2$. So the Hodge index theorem on S yields

$$\pi^*(K_X) \cdot S^2 = \pi^*(K_X)_{|S} \cdot R \geq 2.$$

Set $L := \pi^*(K_X)_{|S}$. If $K_X^3 > 2$, then Lemma 2.5 gives $L^2 > 2$.

In this case, we must emphasize that we only need such a modification π that satisfies 2.4(1) and 2.4(2). Namely, we don't need the normal crossings. Thus we have $\mathrm{Supp}(\triangle) = \mathrm{Supp}(\triangle')$. This property is crucial to our proof.

Now the vanishing theorem gives

$$|K_{X'} + 2\pi^*(K_X) + S|_{|S} = |K_S + 2L|.$$

Because $(2L)^2 \geq 12$, we may apply Reider's theorem again. Assume that $\Phi_{|K_S+2L|}$ is not birational. Then there is a free pencil C on S

such that $L \cdot C = 1$. Note that $R \le 2L$, and that $|R|$ is base point free and $|R|$ is not composed of a pencil. Thus $\dim(\Phi_{|R|}(C)) = 1$. Because C lies in an algebraic family and S is of general type, we have $g(C) \ge 2$. Since $h^0(C, R_{|C}) \ge 2$, the Riemann-Roch theorem on C and Clifford's theorem on C, it easily implies that $R \cdot C \ge 2$. Because $R \cdot C \le 2L \cdot C = 2$, one must have $R \cdot C = 2$. Since

$$2L = S_{|S} + \triangle_{|S} + \pi^*(Z)_{|S}$$

and C is nef, we have $\triangle_{|S} \cdot C = 0$. This implies that $\triangle'_{|S} \cdot C = 0$. Note also that $\triangle''_{|S} = 0$ for general S. We get $(E_\pi)_{|S} \cdot C = 0$. Therefore

$$K_S \cdot C = (K_{X'} + S)_{|S} \cdot C = \pi^*(K_X)_{|S} \cdot C + (E_\pi)_{|S} \cdot C + S_{|S} \cdot C = 3,$$

an odd number. This is impossible because C is a free pencil on S. So Φ_5 must be birational.

Case 2. The case $K_X^3 = 2$.

If $L^2 \ge 3$, then ϕ_5 is birational according to the proof in **Case 1**. So we may assume $L^2 = 2$. By Lemma 2.5, we have $\pi^*(K_X) \cdot S^2 = 2$. Set $C = S_{|S}$. Then $|C|$ is base point free and is not composed with a pencil. So $C^2 \ge 2$. The Hodge index theorem also gives

$$4 = (\pi^*(K_X)_{|S} \cdot C)^2 \ge L^2 \cdot C^2 \ge 4.$$

The only possibility is $L^2 = C^2 = 2$ and $L \equiv C$. On the other hand, the equality

$$4 = 2K_X^3 = K_X^2 \cdot (M + Z) = L^2 + K_X^2 \cdot Z = 2 + K_X^2 \cdot Z$$

gives $K_X^2 \cdot Z = 2$. Take a very big m such that $|mK_X|$ is base point free and take a general member $H \in |mK_X|$. By the Hodge index theorem, $4 = \frac{1}{m^2}(K_X \cdot M \cdot H)^2 \ge \frac{1}{m^2}(K_X^2 \cdot H)(M^2 \cdot H) = 2K_X \cdot M^2$. Thus $K_X \cdot M^2 = 2$ and $(K_X)_{|H} \equiv M_{|H}$. Multiplying it by 2, we deduce that $Z_{|H} \equiv M_{|H}$. Thus $K_X \cdot Z \cdot M = \frac{1}{m} Z_{|H} \cdot M_{|H} = \frac{1}{m} M^2 \cdot H = 2$. So $L \cdot \pi^*(Z)_{|S} = 2$. Since $2C \equiv 2L = \pi^*(2K_X)_{|S} = \pi^*(M + Z)_{|S} = (S + \Delta + \pi^*(Z))_{|S} = C + (\Delta + \pi^*(Z))_{|S}$ and $L^2 = L \cdot C = 2$, we see that

$$0 = L \cdot \Delta = C \cdot \Delta. \tag{3}$$

Thus $K_S = (K_{X'} + S)_{|S} = C + (\pi^*(K_X) + E_\pi)_{|S} = (C + L) + ((E_\pi)_{|S}) = P + N$ is the Zariski decomposition by (3) and 2.4. Denote by $\sigma : S \longrightarrow S_0$ the contraction onto the minimal model. Then $C + L \sim \sigma^*(K_{S_0})$.

Note that $C = S_{|S}$ and $\dim |C| \ge \dim |S|_{|S} \ge 2$ because $|S|$ gives a generically finite map. Assume to the contrary that Φ_5 is not birational. Then neither is $\Phi_{|S|}$. Denote by d the generic degree of Φ_5. Then:

$$2 = C^2 = S^3 \ge d(P_2(X) - 3).$$

Because $d \ge 2$, we see $P_2(X) = 4$ and $d = 2$. As we have shown in Step 1 that

$$|5K_{X'}|_{|S} \supset \text{the movable part of } |K_S + 2L| \supset |C|,$$

we see that $\Phi_{|C|} : S \longrightarrow \mathbb{P}^{h^0(S,C)-1}$ is not birational. On the other hand, we may write

$$2 = C^2 \geq \deg(\Phi_{|C|}) \deg(\Phi_{|C|}(S)).$$

If $h^0(S, C) \geq 4$, then $\deg(\Phi_{|C|}(S)) \geq 2$ and $\deg \Phi_{|C|} = 1$, i.e. $\Phi_{|C|}$ is birational which contradicts the assumption. So $h^0(S, C) = 3$ and $|C| = |S|_{|S}$. Therefore $\Phi_{|C|} : S \longrightarrow \mathbb{P}^2$ is generically finite of degree 2. Let $\Phi_{|C|} = \tau \circ \gamma$ be the Stein factorization with $\gamma : S \to S'$ a birational morphism onto a normal surface and $\tau : S' \to \mathbb{P}^2$ a finite morphism of degree 2. We can write $C = \Phi_{|C|}^* \ell$ with a line ℓ.

For a curve E on S, by the projection formula, $C.E = \ell.\Phi_{|C|*}E$. So E is contracted to a point on S' if and only if E is contracted to a point on $\mathbb{P}^2$ (for τ is finite); if and only if E is perpendicular to $C \equiv \frac{1}{2}\sigma^*(K_{S_0})$ (= half of the pull back of $K_{\overline{S}}$ which is ample on the unique canonical model $\overline{S}$ of S); if and only if E is contracted to a point on $\overline{S}$ by the projection formula again; we denote by E_{all} the union of these E. By Zariski Main Theorem, both $S \setminus E_{all} \to \overline{S} \setminus$ (the image of E_{all}) and $S \setminus E_{all} \to S' \setminus$ (the image of E_{all}) are isomorphisms (so we identify them). Both $\overline{S}$ and S' are completion of the same $S \setminus E_{all}$ by adding a finite set. The normality of $\overline{S}$ and S' implies that the birational morphisms $S \to \overline{S}$ and $S \to S'$ can be identified, so also $S' = \overline{S}$.

Since $\bar{S}$ is normal, Propositions 5.4, 5.5 and 5.7 of [19] imply a splitting

$$\tau_* \mathcal{O}_{\bar{S}} = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{L}$$

where $\mathcal{L}$ is a line bundle. Thus we see that

$$q(S) = q(\bar{S}) = h^1(\bar{S}, \tau_* \mathcal{O}_{\bar{S}}) = 0.$$

Since S is nef and big on X', the long exact sequence

$$0 = H^1(K_{X'} + S) \longrightarrow H^1(K_S) \longrightarrow H^2(K_{X'}) \longrightarrow H^2(K_{X'} + S) = 0$$

gives $q(X) = q(X') = q(S) = 0$. Noting that $\chi(\mathcal{O}_X) < 0$, we naturally have $p_g(X) \geq 2$. By Theorem 3.2, Φ_5 is birational, a contradiction.

Therefore we have proved the birationality of Φ_5. □

Theorem 4.2. *Let X be a projective minimal factorial 3-fold of general type. Assume $d_2 = 2$. Then Φ_5 is birational.*

Proof. **Case 1**. $K_X^3 > 2$.

When $d_2 = 2$, $f : X' \longrightarrow W$ is a fibration onto a surface W. Taking a further modification, we may even get a smooth base W. Denote by C a general fiber of f. Then $g(C) \geq 2$. Pick up a general member S which is an irreducible surface of general type. We may write $S_{|S} \sim \sum_{i=1}^{a_2} C_i$ where $a_2 \geq P_2(X) - 2$. Since $K_X^3 > 2$, we have $a_2 \geq P_2(X) - 2 \geq 3$. Set $L := \pi^*(K_X)_{|S}$. Then L is nef and big. Since $\pi^*(K_X) \cdot S^2 = (\pi^*(K_X)_{|S} \cdot S_{|S})_S \geq 3(\pi^*(K_X)_{|S} \cdot C)_S \geq 3$, Lemma 2.5 gives $L^2 \geq 4$. The vanishing theorem gives

$$|K_{X'} + 2\pi^*(K_X) + S|_{|S} = |K_S + 2L|. \quad (4)$$

Assume that Φ_5 is not birational. Then neither is $\Phi_{|K_S+2L|}$ for a general S. Because $(2L)^2 \geq 10$, Reider's theorem ([27]) tells us that there is a free pencil C' on S such that $L \cdot C' = 1$. Since $2 = C' \cdot 2L \geq C'.S_{|S} = a_2 C' \cdot C \geq 3C'.C$, we have $C \cdot C' = 0$. So C' lies in the same algebraic family as that of C. We may write

$$2L \equiv a_2 C + G$$

where $G = (\Delta + \pi^*(Z))_{|S} \geq 0$ and $a_2 \geq 3$. Since $2L - C - \frac{1}{a_2}G \equiv (2 - \frac{2}{a_2})L$ is nef and big, Kawamata-Viehweg vanishing theorem gives $H^1(S, K_S + \lceil 2L - C - \frac{1}{a_2}G \rceil) = 0$. Thus we get a surjection:

$$H^0(S, K_S + \lceil 2L - \frac{1}{a_2}G \rceil) \longrightarrow H^0(C, K_C + D)$$

where $D := \lceil 2L - \frac{1}{a_2}G \rceil_{|C}$ with $\deg(D) \geq (2 - \frac{2}{a_2})L \cdot C > 1$. Note that $|K_S + 2L|$ can separate different C. If $\deg(D) \geq 3$, then $|K_C + D|$ defines an embedding, and so does $|K_S + 2L|$, a contradiction.

So suppose $\deg(D) = 2$. We now apply Proposition 3.1. Let N' be the movable part of $K_S + \lceil 2L - \frac{1}{a_2}G \rceil$ and let $N = \pi^*(5K_X)_{|S}$. Set $\Lambda := |5\pi^*(K_X)|_{|S}$. As in the proof of Theorem 3.2, we have $\Lambda \supset |N'| +$ (a fixed effective divisor), $|N'|_{|C} = |K_C + D|$, $N' \leq N$ and $\deg(N_{|C}) = 1 + \deg(N'_{|C}) = 2g(C) + 1 = 5$ by the calculation:

$$4 \leq (2g(C) - 2) + 2 = N' \cdot C \leq N \cdot C = 5\pi^* K_X \cdot C = 5.$$

By Proposition 3.1, $\Lambda_{|C} = |N_{|C}|$ gives an embedding. It is clear that $|5\pi^* K_X| \supset |S|$ separates different S, and $|5\pi^* K_X|_{|S}$($\supset$ the movable part of $|K_S + 2L|$) separates different C. Thus Φ_5 is birational. This is again a contradiction.

Case 2. $K_X^3 = 2$.

We first consider the case $L^2 \geq 3$. On the surface S, we are reduced to study the linear system $|K_S + 2L|$. We have

$$2L \sim S_{|S} + G = \sum_{i=1}^{a_2} C_i + G$$

where $a_2 \geq h^0(S, S_{|S}) - 1 \geq P_2(X) - 2 \geq 2$. Denote by C a general fiber of $f: X' \longrightarrow W$. If $a_2 \geq 3$, the proof in **Case 1** already works. So we assume $a_2 = 2$, then $P_2(X) = 4$, and the image of the fibration $\Phi_{|S_{|S}|}: S \longrightarrow \mathbb{P}^2$ is a quadric curve which is a rational curve. This means that $|C|$ is composed with a rational pencil. Assume that $|K_S + 2L|$ does not give a birational map. Then Reider's theorem says that there is a free pencil C' on S such that $L \cdot C' = 1$. We claim that C' is the same pencil as C. In fact, otherwise C' is horizontal with respect to C and $C \cdot C' > 0$. Since C is a rational pencil, $C \cdot C' \geq 2$. Therefore $L \cdot C' \geq 2$, a contradiction. So C' lies in the same family as that of C and $L \cdot C = 1$. Note that $K_S + 2L = (K_{X'} + 2\pi^*(K_X))_{|S} + S_{|S} \geq C$.

So $|K_S + 2L|$ distinguishes different elements in $|C|$. The vanishing theorem gives

$$H^0(S, K_S + \lceil 2L - \frac{1}{2}G \rceil) \longrightarrow H^0(C, K_C + Q)$$

where $Q = \lceil 2L - C - \frac{1}{2}G \rceil_{|C}$ is an effective divisor on C. If $|K_C + Q|$ is not birational, neither is $|K_C|$. So C must be a hyper-elliptic curve. Suppose Φ_5 is not birational. Then Φ_5 must be a morphism of generic degree 2. Set $s = \Phi_5 : X \longrightarrow W_5 \subset \mathbb{P}^N$. Then $5K_X = s^*(H)$ for a very ample divisor H on the image W_5. So

$$5 = 5\pi^*(K_X) \cdot C = 2\deg(H|_{s(\pi(C))}) = 2\deg_{\mathbb{P}^N} s(\pi(C))$$

which is a contradiction. Thus Φ_5 must be birational under this situation.

Next we consider the case $L^2 = 2$. Lemma 2.5 says $2 = \pi^*(K_X) \cdot S^2 = a_2 L \cdot C$. We see that $a_2 = 2$ and $L \cdot C = 1$. We still consider the linear system $|K_S + 2L|$. As above, $a_2 = 2$ implies that $|C|$ is a rational pencil. Since $K_S + 2L \geq C$, we see that $|K_S + 2L|$ distinguishes different elements in $|C|$. By the same argument as above, we have

$$|K_S + 2L|_{|C} \supset |K_C + Q| \supset |K_C|.$$

If Φ_5 is not birational, then neither is $\Phi_{|K_S+2L|}$. This means that C must be a hyper-elliptic curve and Φ_5 is of generic degree 2. With the property that $|5K_X|$ is base point free, we also have a contradiction as in the previous case. So Φ_5 is birational. □

Theorem 4.3. *Let X be a projective minimal factorial 3-fold of general type. Assume $d_2 = 1$. Then Φ_5 is birational.*

Proof. When X is smooth, this theorem was established in [7]. Our result is a generalization.

Taking the modification π as in 2.4, we get an induced fibration $f : X' \longrightarrow W$ and $B := W$ is a smooth curve of genus $b := g(B)$. By Lemma 2.1 of [8], we know that $0 \leq b \leq 1$. Let F be a general fiber of f.

Claim 4.4. *We have*

$$\mathcal{O}_F(\pi^*(K_X)_{|F}) \cong \mathcal{O}_F(\sigma^*(K_{F_0}))$$

where $\sigma : F \longrightarrow F_0$ is the contraction onto the minimal model.

Proof. If $b > 0$, then the movable part of $|2K_X|$ is already base point free by Lemma 2.6. The claim is automatically true.

Suppose $b = 0$. Set $\bar{F} := \pi_* F$. We may write (see 2.4):

$$S = \sum_{i=1}^{a_2} F_i$$

where $a_2 \geq P_2(X) - 1 \geq 3$ and F_i is a smooth fiber of f for each i. Then $2K_X \equiv a_2\bar{F} + Z$. Assume $K_X \cdot \bar{F}^2 > 0$. Then we have

$$\begin{aligned} 2K_X^3 &\geq a_2 K_X^2 \cdot \bar{F} \geq a_2^2 \\ &\geq (P_2(X) - 1)^2 = \frac{1}{4}(K_X^3 - 6\chi(\mathcal{O}_X) - 2)^2 \\ &\geq \frac{1}{4}(K_X^3 + 4)^2. \end{aligned}$$

The above inequality is absurd. Thus $K_X \cdot \bar{F}^2 = 0$ and $\pi^*(K_X)_{|F} \cdot \triangle_{|F} = 0$. Now we apply the same argument as in the proof of Claim 3.3. Thus the claim is true. □

Considering the linear system $|K_{X'} + 2\pi^*(K_X) + S| \supset |S|$, which apparently separates different fibers of f, we get a surjection by the vanishing theorem:

$$|K_{X'} + 2\pi^*(K_X) + S|_{|F} = |K_F + 2\sigma^*(K_{F_0})|.$$

Since F is a surface of general type, $\Phi_{|3K_F|}$ is birational except when $(K_{F_0}^2, p_g(F)) = (1, 2)$, or $(2, 3)$. Thus Φ_5 is birational except when F is of those two types.

From now on, we assume that F is one of the above two types. Then $q(F) = 0$ according to surface theory. By 2.3, one has $q(X) = b$ because $R^1 f_*\omega_{X'} = 0$. Since we may assume $p_g(X) \leq 1$ by Theorem 3.2, $\chi(\mathcal{O}_X) < 0$ and $b \leq 1$, we see that the only possibility is $q(X) = b = 1$, $p_g(X) = 1$ and $h^2(\mathcal{O}_X) = 0$.

Let $D \in |\pi^*(K_X)|$ be the unique effective divisor. Since $2D \sim 2\pi^*(K_X)$, there is a hyperplane section H_2^0 of W' in $\mathbb{P}^{P_2(X)-1}$ such that $g^*(H_2^0) \equiv a_2 F$ and $2D = g^*(H_2^0) + Z'$. Set $Z' := Z_v + 2Z_h$, where Z_v is the vertical part with respect to the fibration f and $2Z_h$ the horizontal part. Thus

$$D = \frac{1}{2}(g^*(H_2^0) + Z_v) + Z_h.$$

Noting that D is a integral divisor, for a general fiber F, $(Z_h)_{|F} = D_{|F} \sim \sigma^*(K_{F_0})$.

Considering the $\mathbb{Q}$-divisor

$$K_{X'} + 4\pi^*(K_X) - F - \frac{1}{a_2}Z_v - \frac{2}{a_2}Z_h,$$

set

$$G := 3\pi^*(K_X) + D - \frac{1}{a_2}Z_v - \frac{2}{a_2}Z_h$$

and

$$D_0 := \lceil G \rceil = 3\pi^*(K_X) + \lceil (1 - \frac{2}{a_2})Z_h \rceil + \text{vertical divisors}.$$

For a general fiber F, $G - F \equiv (4 - \frac{2}{a_2})\pi^*(K_X)$ is nef and big. Therefore, by the vanishing theorem, $H^1(X', K_{X'} + D_0 - F) = 0$.

We then have a surjective map

$$H^0(X', K_{X'} + D_0) \longrightarrow H^0(F, K_F + 3\sigma^*(K_{F_0}) + \lceil(1 - \frac{2}{a_2})Z_h\rceil_{|F}).$$

If F is a surface with $(K^2, p_g) = (2, 3)$, then $\Phi_{|K_F+3\sigma^*(K_{F_0})+\lceil(1-\frac{2}{a_2})Z_h\rceil_{|F}|}$ is birational on F. Otherwise, since

$$\lceil(1 - \frac{2}{a_2})Z_h\rceil_{|F} \geq \lceil(1 - \frac{2}{a_2})(Z_h)_{|F}\rceil = \lceil(1 - \frac{2}{a_2})D_{|F}\rceil,$$

Proposition 2.1 of [9] implies that $\Phi_{|K_F+3\sigma^*(K_{F_0})+\lceil(1-\frac{2}{a_2})Z_h\rceil_{|F}|}$ is birational. Thus Φ_5 is birational. □

Theorems 4.1, 4.2 and 4.3 imply Theorem 1.2.

References

[1] W. Barth, C. Peter, A. Van de Ven, *Compact complex surface*, Springer-Verlag, 1984.

[2] X. Benveniste, *Sur les applications pluricanoniques des variétés de type très général en dimension 3*, Amer. J. Math. **108** (1986), 433-449.

[3] E. Bombieri, *Canonical models of surfaces of general type.* Inst. Hautes tudes Sci. Publ. Math. **42** (1973), 171–219.

[4] Jungkai A. Chen and C. Hacon, *Linear series of irregular varieties*, Algebraic Geometry in East Asia, Japan, 2002, World Scientific Press.

[5] Meng Chen, *On the* **Q***-divisor method and its application.* J. Pure Appl. Algebra **191** (2004), 143–156.

[6] Meng Chen, *On pluricanonical maps for threefolds of general type*, J. Math. Soc. Japan **50**(1998), 615-621.

[7] Meng Chen, *Kawamata-Viehweg vanishing and the quint-canonical map of a complex 3-fold*, Communications in Algebra **27**(1999), 5471-5486.

[8] Meng Chen, *On canonically derived families of surfaces of general type over curves*, Communications in Algebra **29**(2001), 4597-4618.

[9] Meng Chen, *Canonical stability of 3-folds of general type with* $p_g \geq 3$, Internat. J. Math. **14**(2003), 515-528.

[10] L. Ein, R. Lazarsfeld, *Global generation of pluricanonical and adjoint linear systems on smooth projective threefolds*, J. Amer. Math. Soc. **6**(1993), 875-903.

[11] H. Esnault, E. Viehweg, *Lectures on Vanishing Theorems.* DMV-Seminar **20**(1992), Birkhäuser, Basel-Boston-Berlin.

[12] W. Fulton, *Algebraic Topology*, Graduate Texts in Mathematics **153**, Springer-Verlag.

[13] Y. Kawamata, *A generalization of Kodaira-Ramanujam's vanishing theorem*, Math. Ann. **261**(1982), 43-46.

[14] Y. Kawamata, *On Fujita's freeness conjecture for* 3*-folds and* 4*-folds.* Math. Ann. **308** (1997), 491–505.

[15] Y. Kawamata, *Crepant blowing-up of 3-dimensional canonical singularities and its application to degenerations of surfaces.* Ann. of Math. **127**(1988), 93-163

[16] Y. Kawamata, K. Matsuda, K. Matsuki, *Introduction to the minimal model problem*, Adv. Stud. Pure Math. **10**(1987), 283-360.

[17] J. Kollár, *Higher direct images of dualizing sheaves I*, Ann. Math. **123**(1986), 11-42; II, ibid. **124**(1986), 171-202.

[18] J. Kollár, *Shafarevih maps and plurigenera of algebraic varieties*, Invent. Math. **113**(1993), 177-215.

[19] J. Kollár, S. Mori, Birational geometry of algebraic varieties, 1998, Cambridge Univ. Press.

[20] S. Lee, *Remarks on the pluricanonical and adjoint linear series on projective threefolds*, Commun. Algebra **27**(1999), 4459-4476.

[21] S. Lee, *Quartic-canonical systems on canonical threefolds of index 1*, Comm. Algebra **28**(2000), 5517-5530.

[22] T. Luo, *Global* 2*-forms on regular* 3*-folds of general type*, Duke Math. J. **71** (1993), no. 3, 859-869.

[23] T. Luo, *Plurigenera of regular threefolds*, Math. Z. **217** (1994), no. 1, 37-46.

[24] K. Matsuki, *On pluricanonical maps for 3-folds of general type*, J. Math. Soc. Japan **38**(1986), 339-359.

[25] Y. Miyaoka, *The pseudo-effectivity of* $3c_2 - c_1^2$ *for varieties with numerically effective canonical classes*, Algebraic Geometry, Sendai, 1985. Adv. Stud. Pure Math. **10**(1987), 449-476.

[26] M. Reid, *Minimal models of canonical 3-folds*, Adv. Stud. Pure Math. **1**(1983), 131-180.

[27] I. Reider, *Vector bundles of rank 2 and linear systems on algebraic surfaces*, Ann. Math. **127**(1988), 309-316.

[28] E. Viehweg, *Vanishing theorems*, J. reine angew. Math. **335**(1982), 1-8.

[29] P.M.H. Wilson, *The pluricanonical map on varieties of general type*, Bull. London Math. Soc. **12**(1980), 103-107.

[30] S. T. Yau, *On the Ricci curvature of a complex Kaehler manifold and the complex Monge-Ampere equations*, Comm. Pure Appl. Math. **31**(1978), 339-411.

Department of Mathematics, National Taiwan University, Taipei, 106, Taiwan

E-mail address: jkchen@math.ntu.edu.tw

School of Mathematical Sciences, Fudan University, Shanghai, 200433, People's Republic of China

E-mail address: mchen@fudan.edu.cn

Department of Mathematics, National University of Singapore, 2 Science Drive 2, Singapore 117543, Singapore

E-mail address: matzdq@math.nus.edu.sg