# A + B Model in Quantum Geometry

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## GOAL: Toward a Quantum Minimal Model Program

We propose a QMMP (jointly with Y.P. Lee and H.W. Lin) in higher dimensional algebraic geometry.

1. In the orbifold category.
2. Allowing symplectic deformations.
3. Keeping track on the quantum "A + B" model.

A QMMP using only A model (= Gromov-Witten theory, or quantum cohomology QH(X)) was proposed by Yongbin Ruan in 1998.

Difficulty: QH(X) is not functorial under $f: X \to X'$.

## Classical MMP = Mori Program

In 1982, Mori proposed the MMP to classify algebraic varieties of dim > 2 by considering the cone of curves $\overline{NE}(X)$ under $\cap$ with $K_X$:

(1) Let X be a smooth complex variety. If $K_X$ is not nef, then there is an extremal rational curve C in X with $K_X.C < 0$ and a morphism $f: X \to X'$ which contracts all curves C' with $[C'] \in \mathbb{R}[C]$.

(2) If dim X' < dim X then we are done. X is a Mori fiber space over X' with Fano fibers (in principle can be classified).

Otherwise f is birational. Let $Z \subset X$ be the exceptional loci.

(2-1) (Divisorial contraction.) If $\dim Z = \dim X - 1$, then $X'$ has only terminal singularity and we may repeat (1) for the new $X'$ (Kawamata, Kollár).

(2-2) (Small contraction.) If $\dim Z < \dim X - 1$, then $X'$ is too singular to repeat the program. We need to perform a flip $X \dashrightarrow X^+$, which is a surgery from $(Z, C)$ to $(Z^+, C^+)$ to achieve $K.C^+ > 0$ (sign reversed). Then we may repeat (1) for $X^+$.

The termination of flips in dim 3 was proven by Shokurov in 1984.

The existence of flips in dim 3 was proven by Mori in 1988.

Thus the MMP produces $X'$ with nef K, or a Mori fiber space.

Recently, the existence of flips was proven by BCHM (Birkar-Cascini-Hacon-McKernann 2007) in all dimensions. But the termination is still open in dim > 4. (Dim 4 is OK.)

Problems about the minimal models $X'$ of X when dim > 2:

(1) They are singular.

(2) They are not unique.

For (1), in dim 3 the singularities are $cDV/\mu_r$, which are classified by Mori. They can be deformed into cyclic quotients of $C^3$.

For (2), biratinal minimal models are connected by <u>flops</u>. A flop $f: X \cdots> X'$ is a surgery on $C$ with $K_X.C = 0 = K_{X'}.C'$.

This was proven by Kollár–Mori in 1992 in dim 3, and by Kawamata in all dimensions recently. But K–M's methods show that flops in dim 3 preserves the type of Sing(X), moreover the vector spaces H(X) and IH(X) are independent of the choices of minimal models.

DEFINITION: X and X' are K-EQUIALENT if there are birational morphisms $g: Y \to X$, $g': Y \to X'$ such that $g^*K_X = g'^*K_{X'}$ on Y.

In 1997, I proved (in my thesis) that K equivalent projective manifolds X and X' have the same Betti and Hodge numbers. Also birational minimal models are always K equivalent. However,

(1) The canonical (iso-) correspondence $T: H(X) \to H(X')$ has not yet been found for a general K equivalence.

(2) Even if T is found (e.g. for all known smooth flops), T does not preserve the topological ring structures (cup product).

K equivalence conjecture (W- 2000): Such $\mathcal{T}$ exists which induces Def(X) "=" Def(X') and QH(X) "=" QH(X') (quantum cohomology rings).

(1) B-model: In dim 3, invariance of the VHS(X) (variations of Hodge structures) over the complex structure moduli was proven by Kollár-Mori since 3D flops can be performed in flat families.

(2) A-model: In dim 3, invariance of QH(X) over the Kähler moduli space (up to analytic continuations) was first studied by Witten in 1992 and completed by Li-Ruan around 1998.

## Messages from "Nature": String Theory

2D Super-Symmetric Conformal Field Theory predicts the space-time is 10 dim, with 6 extra dimensions in tiny scale. Any geometric model $(X, g)$ of it must satisfy the Vacuum Einstein equation $Ric(g) = 0$. The SUSY requires also that $g$ is a Kähler metric. That is, $X$ is Ricci flat Kähler-Einstein, nowadays called Calabi-Yau manifolds.

THEOREM (YAU 1976): $X$ is Calabi-Yau iff $K_X = -c_1(X) = 0$. Indeed, in each Kähler class $[\omega] \in H^{1,1}(X, R)$ there is a unique Ric flat $g$.

There are $> 10^6$ such X's with different topology which had been found! Each Calabi-Yau 3-fold (real dim = 6) associates "a string theory". However, there is "only one universe". So what's going wrong?

There are two Type II (heterotic) twisted string theories: They are topological quantum field theories (TQFT):

String moduli: $g$ (metric) $\Leftrightarrow$ $(J, \omega)$ (complex, symplectic).

(1) II-A: $J$ fixed, $\omega$ varies $\Rightarrow$ Hilbert space $H_A(X) = \bigoplus H^p(X, \Lambda^q T^*) = H^*(X)$, quantum correlation = Gromov-Witten theory.

(2) II-B: $J$ varies, $\omega$ fixed $\Rightarrow$ Hilbert space $H_B(X) = \bigoplus H^p(X, \Lambda^q T)$, quantum correlation = Kodaira-Spencer theory.

Master Conjecture: Any Calabi-Yau X leads to isomorphic TQFT.

The notion of symmetries in the full Calabi-Yau moduli: All CY 3-folds should be connected to each other via

(1) Crepant (flopping) contractions/resolutions.

(2) Finite Weil-Petersson distance degeneration/smoothing of Calabi-Yau with at most canonical singularities.

These two generate (a) extremal transitions, (b) birational Calabi-Yau's and (c) mirror symmetry near $\infty$ boundaries.

Idea of proof for A-model equivalence in 3D K-equiv:

(1) Decomposition of 3D K-equiv into flops. (OK by Kollár-Mori.)

$$X \dashrightarrow X_1 \dashrightarrow X_2 \cdots \dashrightarrow X'.$$

(2) Symplectic deformation of any smooth 3D flop into copies of Atiyah $P^1$-flop: In the K-equiv diragram,

$$Z = P^1, \quad N_{Z/X} = O(-1)^2, \quad Y = Bl_Z(X).$$

(OK by Mori's classification and Freidman's study on ODP.)

For each $P^1$-flop, $T = g'_* g^*: H(X) \to H(X')$ is canonical in the sense that $T$ preserves the Poincaré pairing.

(3) Quantum corrections: Let a, b, c $\in H^2(X)$. Then

$$\Delta := (Ta.Tb.Tc)^X - (a.b.c)^{X'} = (a.Z)^X(b.Z)^X(c.Z)^X.$$

Witten and Aspinwall-Morrison found that each degree d map f: $P^1 \to Z = P^1 \subset X$ has contribution by $1/d^3$. Hence the quantum 3-point function (virtual int. number via f) is given by

$$\langle a, b, c\rangle^X = \Sigma_{d\in N}(a.Z)(b.Z)(c.Z)q^d = \Delta q/(1 - q).$$

Here $q = \exp(-2\pi(\omega.Z))$ depends on the Kähler variable $\omega$. Since $T(Z) = -Z'$, we have $Tq = 1/q'$. Hence

$$\langle Ta, Tb, Tc\rangle^{X'} - T\langle a, b, c\rangle^X = \Delta(q'/(1 - q') + Tq/(1 - Tq)) = -\Delta.$$

Note: The convergent radii have no intersection!

(4) Degeneration of GW theory into local models. (OK by Li-Ruan.)

We (LLW) generalize this "analytic continuation" of QH to ordinary $P^r$-flops in all dimensions: Namely

$$Z = P^r, \quad N_{Z/X} = O(-1)^{r+1}, \quad Y = Bl_Z(X),$$

as well as the family case* when $Z = P_S(F) \to S$ is a $P^r$ bundle.


[1] LLW; Flops, motives and inv of quantum rings, Annals.

[2] FW; Motivic and quantum inv under stratified Mukai flops, JDG.

[3] ILLW; Inv of GW theory under simple flops (all genera).

[4] LLW; Quantum inv under flop transitions (CY flops), Yau's 59th.

[5]* LLW; Inv of quantum rings under ordinary flops.


NEW TOOLS: Reconstruction, quantization and re-normalization.

B model: The $g = 0$ theory $\equiv$ VHS.

(1) For $f: X \to S$ a smooth family, $R^k f_* \mathbb{C} \to S$ is a VHS of weight $k$ with Gauss-Manin connection $\nabla^{GM}$, Hodge filtration $\{F^p\}$ and flat (integral) structure $R^k f_* \mathbb{Z}$. Griffiths trans: $\nabla F^p \subset \Omega_S(F^{p-1})$.

(2) For family of CY k-folds, rk $F^k = 1$ with local frame $\Omega$. The periods integral $\int_\Gamma \Omega$ satisfies a Picard-Fuchs equation.

A model: The $g = 0$ theory $\equiv$ (QH, $*$).

(1) Let $H = H_A = H^*(X)$. The tangent bundle $TH = H \times H$ has a Dubrovin connection $\nabla_a := D_a - z^{-1}\, a *_t$ $(t \in H)$.

WDVV Equation: $\nabla$ is flat $\Leftrightarrow$ $*$ is associative.

(2) If X is toric, then "QH" also satisfies a Picard-Fuchs Eq.

(1) Mirror Symmetry: Up to SUSY, the eigen-spaces of super charges are exchanged. This predicts that for a CY 3-fold X, there exists another CY 3-fold X' s.t. $h^{1,1}(X) = h^{2,1}(X')$ and $h^{2,1}(X) = h^{1,1}(X')$.

Conjecture (Candelas 1990, BCOV 1993): $A(X) = B(X')$ and $B(X) = A(X')$ in the large complex/Kähler structure limits.

For quintic CY: $X = (5) \subset P^4$, the $P^1$ counting problem on X can be solved by the Picard-Fuchs ODE on X'. (Givental, LLY 1997.) This can be generalized to toric complete intersections. For examples, the dim $k - 2$ Calabi-Yau hyper-surfaces: $CY_k = (k) \subset P^{k-1}$.

(2) Birational Calabi-Yau's: $A(X) = A(X')$ and $B(X) = B(X')$.

(3) Extremal Transitions: Let $f: X \to X_0$ be a crepant contraction and $X_t$ $(t \in S)$ be a smoothing of $X_0$. Denote a general $X_t$ by $X'$. The process $X$ to $X'$ is also a K-equiv (up to deformation). This is well defined without the Calabi-Yau assumption.

(3-1) $X$ has more $H^{1,1} = H^1(\Omega^1)$ than $X'$, since $X$ contains the extremal ray $L$ under $f$. Thus $A(X) > A(X')$.

(3-2) $X'$ has more $H^{2,1} = H^1(T)$ than $X$, since $X'$ contains the vanishing cycle $\Gamma$ of the degeneration. Thus $B(X) < B(X')$.

CONJECTURE: $A(X) + B(X) = A(Y) + B(Y)$.

LOCAL EXAMPLES: Consider the dim k hyper-surface $X_0 \subset C^{k+1}$:

$$x_0^k + x_1^k + \cdots + x_k^k = 0$$

with $p = 0 \in X_0$ being an ordinary k-fold singularity. The blow-up $f: X = Bl_p(X_0) \to X_0$ is crepant with exceptional divisor

$$E = (k) \subset P^k, \ N_{E/X} = O(-1)|_E.$$

The local structure of $E \subset X$, namely the germ $(E, X)$ is equivalent to $P^k$ "cut out" by the rank 2 vector bundle:

$$V_k = O(k) \oplus O(-1) \to P^k.$$

$X_0$ can be smoothed into a flat family $M \to \Delta$ with general smooth fiber $X' = M_t$. The semi-stable reduction $\pi: W \to \Delta$ is used to compare X and X' since $W_t = X'$ and $W_0 = X \cup E'$ for some Fano E'.

## Quantum Transition from A to B:

The Gromov-Witten extremal function $f(a) = \Sigma_{d\in N} \langle a\rangle_{dL}\, q^{dL}$ attached to the extremal ray $L \in NE(X)$ can be calculated, using the quantum Serre duality principle, by the bundle

$$V_k^+ = O(k)\oplus O(1) \to P^k.$$

This is in turn reduced to $O(k) \to P^{k-1}$, the Calabi-Yau $CY_k$!

Where is the Picard-Fuchs operator $P_k$ for $f(a)$?

Since $\dim CY_k = k - 2$, we must have $\deg P = k - 2$. But $\dim X' = k$. It must be the case that there is a "sub-VHS of $R^k\pi_*C$ of weight $k - 2$" which starts at $\Omega \in H^{n-1,1} = H^1(X', T)$. Let $\Gamma$ be the vanishing cycle along $\pi$, then $P_k$ is the Picard-Fuchs op for $\int_\Gamma \Omega$.

## BACK to the QMMP:

1. Allowing deformations, it seems that the MMP (at least for dim 3) can be performed purely in orbifolds with only divisorial contractions ($K \searrow$) and flops ($K$ equiv) (Chen–Hacon).
2. GW theory has been extended to orbifolds (Chen–Ruan, CCTY).
3. The A + B invariance of orbifold flops is as expected.
4. For divisorial contraction $f: Y \to X$, the key claim is that it is precisely the A + B model which satisfies the change of variable formula (functoriality, W– 2000).
5. The A + B "should be" an extension of flat bundles. THANKS.